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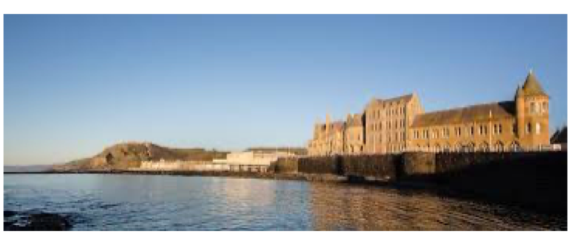

Contents

Abstracts

Factorisation of the matrix polynomial: what does exact factorisation mean?

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Let $A(t)$ be a matrix function from the matrix Wiener algebra $W^{p\times p}(\mathbb{T})$ that is invertible on the unit circle $\mathbb{T}$. The representation

$$A(t) = A_+(t)D(t)A_-(t), \quad t \in \mathbb{T}$$

is called *a left Wiener–Hopf factorisation* of $A(t)$. Here $A_\pm(t) \in GW_\pm^{p\times p}(\mathbb{T})$, where $GW_\pm^{p\times p}(\mathbb{T})$ are the groups of invertible elements of the subalgebras $W_\pm^{p\times p}(\mathbb{T})$, $D(t)$ is the diagonal matrix $D(t) = \operatorname{diag}\left[t^{\lambda_1}, \dots, t^{\lambda_p}\right]$, where integers $\lambda_1 \geq \dots \geq \lambda_p$ are *the left partial indices* of $A(t)$.

This problem has extensive applications in different areas of mathematics, mechanics and physics. Unfortunately, there are two main obstacles for its wider applications: there does not exist a general method for explicit factorisation of matrix functions, while possible factorisation is not always stable in other words the problem is ill-posed.

As a result, even if an explicit algorithm for factorisation of a class of matrix functions has been developed, its implementation into a software requires approximate calculations, which may not be properly realised due to the aforementioned instability.

To overcome this issue, obe can utilise exact calculations, that is, calculations in rational arithmetic. Unfortunately, this is not always possible.

In the talk, for the class of Laurent matrix polynomials, a criterion for the existence of exact solution of the factorisation problem is found [1]. Based on the well-known explicit algorithm [2], an exact solution is constructed and implemented as a package ExactMPF in Maple [1, 3, 4]. Numerical experiments are presented.

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Elastic solids moving along frictionless constraints via configurational forces

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The concept of configurational force has been introduced in solid mechanics by Eshelby [1] to model the position change of defects within solids. Throughout the years this concept has been exploited to provide the motivation for the motion of dislocations and for crack propagation. More recently, configurational forces have been shown to be present in variable-length structural systems [2], realized with frictionless sliding sleeve constraints [3, 4].

Within a quasi-static framework, it is shown that a configurational force acting on a hyper-elastic solid can be interpreted as the resultant reaction realized by frictionless rigid constraints, which allows a relative motion between the solid and the constraints. A simple analytical expression for the configurational force acting on a hyper-elastic solid is derived through energy and perturbative approaches. The analysis is extended to the dynamic regime by solving the Lagrange's equations obtained under the simplifying assumption of a piecewise constant deformation gradient tensor within the solid. Finally, the semi-analytical prediction is validated for a Mooney-Rivlin material through comparison with numerical results obtained from a finite element analysis.

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Development of thermal shock protocol of experiment of carbon-based refractory materials

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Ladle shrouds, tundish nozzles, slide gates, and other flow control refractory materials play an important role in ensuring good production and quality of products in the continuous casting of steel; however, their chemical, mechanical, and thermal degradation and wear mechanism can lead in some cases to disastrous social, environmental, and financial effects in the steel-making plants. Therefore, it is both important and challenging to understand and characterize their behavior at a laboratory scale. This study aims to show the primary steps of investigating the thermal shock resistance of flow control products and mainly Alumina-Carbon based mixes. The adopted approach for the novel testing protocol is to use induction as a heating source to produce an ascending thermal shock and assess the resultant damage. The preliminary results confirm the feasibility of the test and the ability to measure strain evolution and crack initiation as a function of time and temperature. The microstructure of the tested specimen will then be inspected to allow a complete understanding of the behavior of the tested specimens under thermal shock test and permits the prediction of their behavior in steel factories.

Homogenization of chemically heterogeneous adhesive contact — Asymptotic models

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In the overwhelming majority of studies on adhesive contact published to date, (i) the condition of axisymmetric contact geometry is employed, and (ii) the interface energy (specific work of adhesion) is assumed to be spatially constant. Considering the first issue, it should be noted that analytical theories of rough adhesive contact, while considering non-axisymmetric macro contact geometry, still utilize axisymmetric models of contact with adhesion at the micro scale (at asperity level). A recent progress in solving contact problems with chemically heterogeneous interface energy was achieved by Lyashenko *et al.* [1] and Sanner and Pastewka [2] who used different approaches — namely the method of dimensionality reduction (MDR) and the crack front perturbation method. Still, the Hertzian axisymmetric contact geometry was adopted in those studies following the Johnson–Kendall–Roberts (JKR) theory of adhesive contact, whereas the inconstancy of the interface energy induces deviation of the contact area from a circular domain.

In the present study, contact of elastic solids with heterogeneous interface energy of adhesion is considered. The known results on the asymptotic analysis of the quasi-static crack propagation [3, 4] along with the Griffith-type approach (based on the energy balance) and the Irwin-type approach (based on the concept of the contact stress intensity factor) are applied to generalize the JKR theory of adhesive contact. Under the assumption of slightly heterogeneous adhesion, the effective interface energy is determined depending on the approach used. The asymptotic equivalence between the three derived models, including that based on the mixed Griffith–Irwin-type approach, has been established. The leading-order asymptotic approximation for the pull-off force is derived. The cases of the ray-shaped and periodically-patterned adhesion are considered in detail.

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Flutter instability & oscillatory instabilities in piecewise-smooth elastic structures

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Flutter instability caused by follower loads has become a reality after the invention of the "freely-rotating wheel device" by Bigoni and Noselli [1], of the "flutter machine" [2, 3], and of the device to generate Reut-type loads [4]. Further research has proven that flutter instability, Hopf bifurcation, dissipation instabilities, and Ziegler paradox are all possible in conservative systems, thus disproving an erroneous belief continuing since at least 50 years [5]. The last part of the talk addresses a new type of flutter instability generated by the "fusion" of two structures which are separately stable, but become unstable when joined together. The analysis of instability involves here the treatment of a discontinuity in the curvature of a constraint [6].

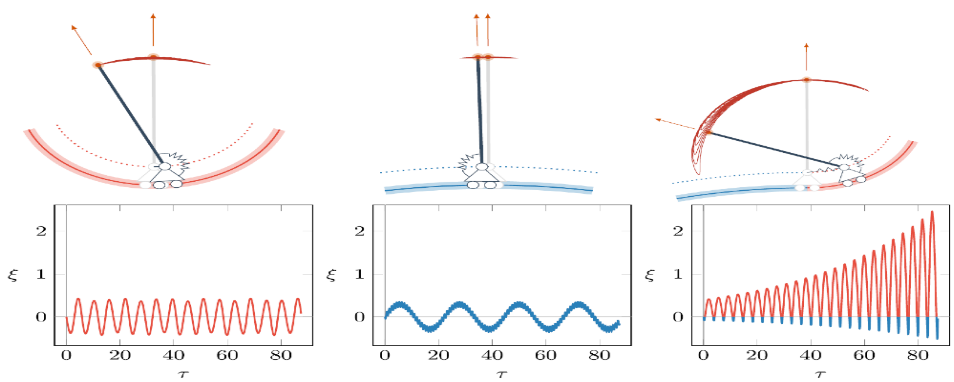

Figure 1: Two stable smooth subsystems with positive and negative curvature of a sliding constraint (upper part: left and centre) and the fusion of these two structures, namely, a compound non-smooth structure displaying instability (upper part: right), although the two 'components' are stable. The tensile force acting at the free end of the rods is tangentially follower and the same for all three structures, lying well below the critical load for instability in the case of the two smooth 'component systems'.

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On some generalizations of the Janashia–Lagvilava spectral factorization method

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We briefly describe the Janashia–Lagvilava method of matrix spectral factorization and provide its different generalizations. In particular, we extend the method to J-spectral factorization and to the factorization on the real line. In addition, we illustrate a contribution of the method to constructing orthogonal matrices over finite fields, which may seem somewhat unexpected at first glance.

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Fracture propagation as a standard dissipative process: application to hydraulic fractures in energy storage materials

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Li-ion batteries based on solid electrolytes (SEs) are extremely promising yet suffer from challenges such as: poor ionic conductivity compared to liquids, high grain boundary resistances, instability in contact with Li. Hybrid electrolytes (HEs) might be the answer: they are a combination of ceramic SEs with liquid electrolytes. Although HEs benefit from each component's potential, the penetration of fluid into the inevitable cracks in ceramic brittle electrolytes may rise hydraulic fractures concerns.

Recent publications framed the problem of three-dimensional quasi-static crack propagation in brittle materials into the theory of standard dissipative processes [1]. Variational formulations, stated therein, characterize the three-dimensional crack front quasi-static velocity as the minimizer of constrained quadratic functionals. An implicit in time crack tracking algorithm, that computationally handles the constraint via the penalty method algorithm, was developed and implemented in [2].

Although the theoretical setting is sound, the derived crack tracking methods suffered from a major drawback that limited the interest in the method to its theoretical content. Specifically, the need of still currently unavailable accurate approximations for weight functions made the approach of minor interest from the numerical standpoint. Such a drawback was overcome in [3], where a viscous regularization of the fracture propagation in brittle materials as a standard dissipative process was formulated. Rate-dependency provided a simple and accurate approximation of the crack front velocity, thus allowing to formulate effective crack tracking algorithms.

That idea is further developed here to model hydraulic fracture processes in energy storage systems. Whereas classical fracking problems [4] are viscous dominated, the propagation of fractures in wet brittle electrolytes clearly shows a lag between fluid and crack tips. A novel set of integral and differential equations are here proposed, which are capable to model the evolution of the fluid lag and of the crack advancing in a straightforward way.

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Development of Thermoplastic Constitutive Models for Refractory Ceramics in Wide Temperature Range

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Refractory components are fundamental in the steel industry for the manipulation of liquid metal, but they have a very short life cycle and cannot easily be recycled, leading to a great waste of material. To sustain critical temperature conditions and high thermal gradients, refractories are made of ceramic materials whose mechanical behavior at service temperature can be brittle. To avoid unexpected brittle fractures these components are replaced after a fixed number of cycles or at the first appearance of a visible crack. Modeling and simulation can greatly enhance the design of these components to maximize material usage and working life avoiding early disposal. The goal of this work is to provide a new computational framework able to describe the complex behavior of refractories under working conditions, by combining constitutive modelling and fracture mechanics. This combination is made possible by the use of the phase field variational theory for fracture mechanics, which allows the analysis of damage in a solid as a scalar variable at each material point. A new constitutive model is proposed by combining the Bigoni-Piccolroaz yield criterion with polynomial hardening functions based on the Bezier curves. The asymmetric tension-compression behavior of ceramic materials is described by a spectral elastic strain energy split. A new enrichment of the crack driving force with a term taking into account plastic hardening is proposed. The model is implemented on a Finite Element framework and validated using the experimental results of thermal shock experiments provided by the industrial partner.

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A general approximate solution for the slightly non-axisymmetric normal contact problem of layered and graded elastic materials

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Contact mechanics of layered and functionally graded elastic materials have received a lot of interest in the framework of modelling tribological properties of, for example, articular cartilage, coatings, biomaterials, or soil. As the — either discrete or continuous — material inhomogeneity severely complicates (or, to put it bluntly, apart from some special cases like the power-law graded elastic half-space, inhibits) an exact closed-form solution of corresponding contact problems, analytical contact solutions of layered materials are often in approximate or asymptotic form. Because of the approximate character of these solutions, their predictions should always be checked against the results of rigorous numerical simulations, a step, which was simplified massively recently for the case of a single elastic layer, with the publication of a boundary element method (BEM) formulation of the corresponding normal contact problem.

An ingenious principle to solve contact problems stems from Mossakovski and later Jger. It is based on the observation that the incremental difference between two subsequent contact configurations can be understood as an infinitesimal indentation by a flat punch. Hence, the general normal contact can be thought of as a series of incremental flat punch indentations, and therefore, the solution procedure is split into two tasks: the determination of the relation between indentation depth and contact region (which encodes the correct series of flat punches to exactly reproduce the original contact), and the solution of the corresponding flat punch problems. For the case of axisymmetric indentation of an elastic half-space, both tasks are easily solvable, which leads to the famous solution that is often attributed to Sneddon, although it originated a lot earlier. Very recently, Popov published an approximate analytical solution for the slightly non-axisymmetric version of this contact problem, which has proven (by comparison with rigorous numerical solutions) to give very satisfactory results even for contact geometries that are far from axial symmetry. Popov's solution rests on two fundaments: On the one hand, Barber's extremal principle that the correct contact region at a given indentation depth maximizes the corresponding total normal force; and, on the other hand, Fabrikant's approximation for the pressure distribution under a flat punch of arbitrary (compact) planform.

As we will show, Barber's principle applies to any elastic normal contact problem (at least, with compact contact regions), which can be thought of as a series of flat punch indentations. Moreover, the "essence" of Fabrikant's approximation is to "scale" the axisymmetric pressure distribution under a cylindrical flat punch to the asymmetric arbitrary planform; a procedure which also can be executed very generally, if the indented material is sufficiently isotropic. Hence, both fundaments of Popov's approximate solution for the homogeneous half-space can be generalized for layered or functionally graded materials (albeit in a slightly less rigorous sense, as will be discussed in the talk), and it is thus expected that the solution itself can be generalized for the application with layered media, as well. This is the topic of the talk.

The obtained approximate solution is, of course, compared to rigorous BEM-based numerical results. As a special, fully analytical, case, the indentation of a power-law graded elastic half-space is considered in detail.

Effects of different loadings on bifurcation of a coated elastic disk

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Nowadays several technologies involve coating of bulk materials with a thin layer made up of another material, so that the physical properties of the system can be properly designed and enhanced, [1, 2]. From a mechanical point of view, a coating layer diffuses the load on an attached solid in a non-local way and strongly affects the mechanical response and failure/buckling mechanisms of the coated object. Cylindrical bodies, rings and arches can undergo instability when loaded by an externally applied pressure and display a characteristic oval bifurcation mode, [3–5]. However, the situation is completely modified by the presence of an inside material [6, 7] and depending on mechanical and geometrical properties of the coating/disk system, different situations are produced and at the bifurcation, critical modes can display wavy patterns, Fig. 1.

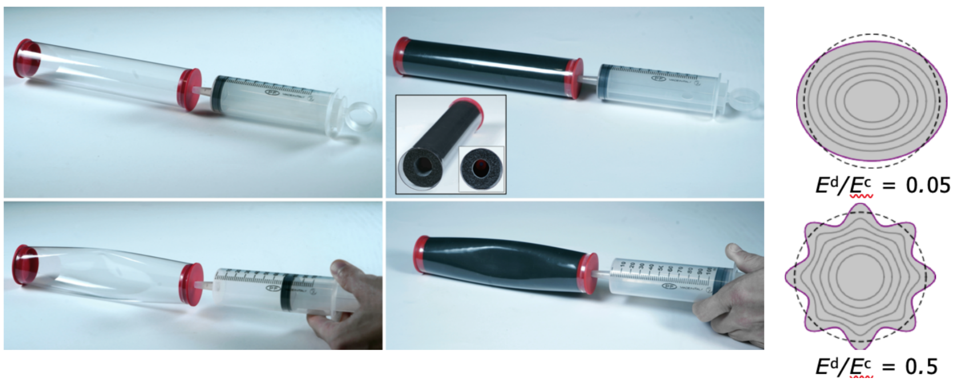

Figure 1: Buckling induced by pressure on an empty thin cylindrical shell (left) and on a shell enclosing a material (middle). The different ratio between Youngs modulus of the disk (E^d) and of the coating (E^c) produces various bifurcation patterns which can be different from the mere oval modes of rings (right).

The case of an elastic thin layer perfectly bonded to an elastic disk is analyzed when loaded with uniform radial load before bifurcation. More specifically, the inextensible and unshearable rod model for the coating is presented and used for modelling a thin film attached to the boundary of a circular elastic and isotropic disk. Hence, the rod is acting as a coating for the disk where the axial inextensibility enforces an isoperimetric constraint for the disk which is constrained to maintain its perimeter during the deformation process. The governing equations for the coating/disk system are formulated for three types of applied external load [8–10], and their effect on the bifurcation is analyzed. Relying on the complex potential formalism [11,12], the problem is solved on the disk and a closed-form solution for the critical pressure is produced while Kolosov-Muskhelishvili complex potentials allow the determination of elastic fields on the boundary and within the disk.

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On the partial indices of piecewise constant matrix functions induced from Fuchsian system

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In the talk we discuss relation between the Fuchsiam systems of differential equations and Riemann–Hilbert boundary value problem with piecewise con- stant transition functions on the Riemann sphere. In particular, we prove that, for any Fuchsian system there exists a rational matrix function whose partial indices coincide with the splitting type of the canonical vector bundle induced from the Fuchsian system. From this we obtain solution of the Riemann–Hilbert boundary value problem for piecewise constant matrix func- tion in terms of holomorphic sections of vector bundle and we give algorithm for calculating L_p-partial indices for the piecewise constant matrix functions induced from monodromy representation of the Fuchsian system. The results given in the talk based on the work [1] and [2].

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Wave propagation in a triangular lattice with discrete sources placed on line segments

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We investigate the propagation of time-harmonic waves through a triangular lattice with sources located on line segments. Specifically, the study focuses on the discrete Helmholtz equation, where the wave number k lies within the range of $(0, 2\sqrt{2})$, and the input data is prescribed on finite rows and columns of lattice sites without resorting to complex wave numbers. Similar to the continuum theory, the notion of a radiating solution is introduced, establishing a unique solvability result and Green's representation formula employing difference potentials. Furthermore, we apply a numerical computation method that demonstrates efficiency in solving problems related to the propagation of left-handed 2D inductor-capacitor metamaterials.

Acknowledgment: The work was supported by the Shota Rustaveli National Science Foundation of Georgia (SRNSFG) [grant number: FR-21-301, project title: "Metamaterials with Cracks and Wave Diffraction Problems"].

Adhesion instability and edge effect in elastic contact: Numerical simulation using the FFT-assisted BEM

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The talk will present two contact problems: (1) adhesion instability of rough surfaces and (2) edge effect in contact of elastic quarter space, which are numerically studied using the fundamental equation-based boundary element method (BEM) (sometimes referred to as the semi-analytical method). To accelerate the calculation of the surface displacement as well as the stress distribution of the half-spaces, the technique of the Fast-Fourier-Transform (FFT) is applied in the simulation. In the case of adhesive contacts, the stress criterion proposed in [1] is used to determine the detachment of elements at the contact boundary, which is derived from the comparison of the elastic energy stored in the element with its surface energy. This approach has been extensively validated by comparison with analytical solutions. In particular, it can accurately reproduce the classic Johnson–Kendall–Roberts (JKR) results for Hertzian contact.

The first contact problem is rough contact. The "adhesion instabilities" of rough surfaces at the contact boundary cause energy dissipation similarly to the elastic instability mechanism. This leads to different effective works of adhesion when the contact area expands and contracts. This effect is interpreted in terms of "friction" to the movement of the contact boundary. Different values for the effective work of adhesion when the contact area expands and contracts (adhesion hysteresis) are a direct consequence. Pure rolling is essentially a normal contact problem because the surfaces at the leading edge are approaching each other in the normal direction and on the rear edge they separate in normal direction, both without any tangential movement [2].

In the second part the edge effect will be presented where the contact region is on or very close to a sharp edge or corner. The contact of an elastic quarter- or eighth-space is studied under condition that the movement of the side surface of the quarter-space is constrained: it can slide freely along the plane of the side surface but its normal movement is blocked. This is the simple case of Hetényi's contact problem (1970): one can obtain its solution easily by applying an additional mirror load to an elastic half-space. Depending on the position of the indenter relative to the side edge, different contact behavior is observed. In the case of adhesive contact, the force of adhesion first increases with increasing the distance from the edge of the quarter-space, achieves a maximum and decreases further to the JKR-value in large distance from the edge. The enhancement of the force of adhesion compared to the half-space contact is associated with the pinning of the contact area at the edge. The maps of the force of adhesion and their analytical approximations are provided [3].

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Waveguides subjected to gyroscopic forces. Temporal modulation. Imperfect interfaces

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We discuss elastic waveguides, characterised by the dynamic coupling of displacement compo-nents due to gyroscopic forces. In the governing equations, such coupling appears to be formal-ly similar to the Lorentz force observed in problems of electromagnetism. Gyroscopic chirality in metamaterials waveguides influences the wave dispersion, formation of stop bands and pass bands, as well as wave localisation. The recent work [1, 2] includes the deriva-tion of the mathematical model and the analysis of the dynamic response of multi-scale chiral systems. For temporally inhomogeneous media the coefficients in the governing equations may change with time. The book [3] presents the mathematical theory of dynamic materials, and the papers [4–6] study wave patterns in structures with spatial and temporal interfaces. With causality always in place for physical processes, interesting wave phenomena are observed when a wave is split at a temporal interface. The work [7] has addressed the energy balance relations for waves in a spatio-temporal material composite. The lecture is based on the results of the paper [8], which includes modelling of transient wave phenomena in structures that have stratification in the temporal dimension. The emphasis is on the analysis of temporal modulation of frontal waves. In particular, special regimes have been identified where the solution shows growth in time. Imperfect interfaces, across which the displacements are discontinuous, are considered in the vector case of chiral elastic systems. Wave patterns are analysed for different types of materials and different initial conditions.

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The functional analytic approach for degenerating boundary value problems

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The functional analytic approach (FAA) is a method whose goal is to represent the solutions of perturbed boundary value problems in terms of real analytic maps and known functions of the perturbation parameters. In this talk, we present some recent applications of the FAA to the study of boundary value problems with degenerating boundary conditions. First we consider the behavior of the solutions of a mixed problem for the Laplace equation in a domain Ω. On a part of the boundary $\partial\Omega$ we consider a Neumann condition, whereas in another part we consider a nonlinear Robin condition which depends on a positive parameter δ in such a way that for $\delta = 0$ it degenerates into a Neumann condition. We describe what happens to the solution $u(\delta, \cdot)$ as $\delta \to 0$ by means of representation formulas in terms of real analytic maps (see [4]). Then we consider the case when the part of the boundary where the Robin condition is imposed shrinks to 0. As we shall we will need to consider separately the case of dimension $n = 2$ and $n \geq 3$ (see [3, 5]). Finally, we will show how some of the above described results can be extended to the case of a degenerating Robin traction problem for the Lamé equations in a periodically perforated domain (see [1, 2]).

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Polyconvexifcation of isotropic functions

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The aim of numerous problems in non-linear elasticity is to find the minimizers of the functional

$$I : W^{1,p} \to \mathbb{R} \cup \{\infty\}; \quad u \mapsto \int_\Omega W(\nabla u(x)) - f(x) \cdot u(x) dx.$$

For the minimisation of I in the context of elasticity, the concept of polyconvexity (of W) becomes crucial. In particular, when the function W is not polyconvex, its lower polyconvex envelope can be utilized for relaxation and numerical simulations. We introduce a new criterion for determining the polyconvexity of isotropic functions, which significantly reduces the number of dimensions compared to the original definition [1]. Using this criterion, we present numerical computation methods that enable the calculation of polyconvex envelopes in three dimensions.

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Dynamic characterisation of mesoscale discrete systems: from 1D flexural waveguides to the generalised Rayleigh beam

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The dynamic behaviour of asymmetric flexural systems, involving a master beam attached to a non-periodic collection of flexural resonators, is discussed [1]. The resonators couple longitudinal and flexural responses of the master beam. Its response is described via Greens functions, with intensities determined from an algebraic system embedding interactions of individual resonators. For infinite periodic waveguides, we derive an effective model called the generalised Rayleigh beam that supports flexural-longitudinal wave coupling and is efficient in regimes not typically encountered in homogenisation [2].

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Transmission conditions across a thin thermoelastic interphase

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Refractory devices composed of ceramics are employed wherever there is contact with molten metals as in crucible, filters, furnaces or systems for flow control. These devices are composed as a mixture of two or more materials, and thus have varying elastic and thermal properties. As these devices operate at very high temperatures, the mixture undergoes thermal expansion at different rates, which can lead to crack initiation, and ultimately the failure of the materials. The work mainly focuses on the development of transmission conditions across a thin thermoelastic interphase. The interphase is modeled as a thin, "spring-type" elastic layer of constant thickness, whose properties are different from those of the surrounding media, and is subjected to a high thermal load. An asymptotic approach is used to derive the non-linear transmission conditions across the interphase, and results are analyzed using finite element methods.

Challenges in tracing the crack front of a hydraulic fracture in inhomogeneous media

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This talk will discuss the peculiarities of modelling hydraulic fracture propagating within a media with variable toughness. Numerous simulations have been performed utilizing our extremely accurate and effective in-house built time-space adaptive solver, which can obtain solutions for any of the 1D HF models (PKN, KGD, Radial, 3PD, non-local PKN) with arbitrary (fixed) fluid rheology, leak off law and pumping regime. The solver uses the crack opening and the fluid velocity as the basic unknowns in contrast to the conventional crack opening and fluid pressure pair [1,2]. We analyse the KGD and Radial HF models in an elastic material characterised by a periodic toughness distribution. Recently [3, 4], we have proposed an averaging-based approach that is dependent not only on the material but also on process dependent parameters. Their definition comes from temporal averaging (in contrast to the spacial one). As a result, all introduced measures rely also on the instantaneous crack speed. Such temporal average approach is general in its nature (not specific to HF), and can be used in analysis of any stable fracture propagation process. The simulations performed for both KGD and radial crack settings have allowed us to validate the temporal-averaging concept showing, among others, how the effective (averaged) toughness approaches its maximum value when the crack is sufficiently long, as was elegantly claimed by Dontsov and Suarez-Rivera [5].

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Hydraulic fracture growth through layers of varying toughness

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The hydraulic fracturing technology consists of growing tensile fractures via the injection of a viscous fluid into a solid. Engineered Hydraulic Fractures (HFs) are extensively used to enhance the fluid production from geological reservoirs but also to measure the minimum in-situ stress in deep rock formations. In most sedimentary basins, the stress conditions are such that hydraulic fractures propagate on a vertical plane with negligible fluid lag, and they interact with layer-like heterogeneities. The vertical growth of HFs, outside the formation of interest where the injection is performed, is a critical criterion determining the feasibility of the technique [1]. In particular, the ratio between the vertical and horizontal extent of fracture propagation largely depends on the natural variation of the in-situ confining stress and mechanical properties between the layers [2, 3]. In this work, we quantify the sole effect of fracture toughness variation on vertical HF containment. We consider the propagation of a HF growing driven by the injection of a viscous fluid through multiple layers of piece-wise homogeneous toughness. The instantaneous velocity of propagation at a generic position along the front oscillates around an average velocity that evolves with time. We demonstrate that the ratio between the vertical and the horizontal average velocities tends to two different asymptotic limits [4]. In one case this limit is zero, resulting in a "late time" contained vertical growth. In the other case the propagation becomes self-similar (see Fig. 1).

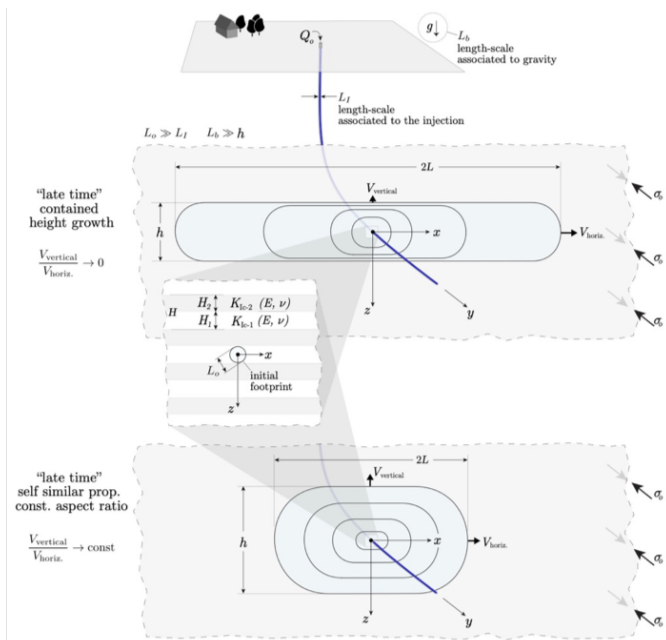

Figure 1: Two "late time" asymptotic footprint evolutions for a hydraulic fracture propagating through multiple layers of piece-wise homogeneous toughness.

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Water-wave diffraction and dissipation by floating or submerged poro-elastic plates

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We discuss the application of the Wiener–Hopf method to linear water-wave interactions with horizontal thin plates, in particular porous, elastic and poro-elastic plates, in a unified framework. It is shown how the solution for a single plate can be computed straightforwardly using Cauchy-type integrals, which avoids the need to find the roots of the highly non-trivial dispersion equations. Furthermore, the method is illustrated with some numerical computations, focusing on the evolution of an incident wave pulse which illustrates the existence of two transmitted waves in the submerged plate system. The effect of the porosity is shown to influence the shorter-wavelength pulse much more strongly than the longer-wavelength pulse. Finally, the extension to multiple plates will be discussed.

Assessment of fracture properties through wedge splitting test and inverse analysis

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The present study investigates the assessment of mechanical parameters, with a focus on specific fracture energy, of ceramic materials using data collected from a Wedge Splitting Test (WST) and Inverse Analysis (IA). Developed IA procedure is centered on the minimization of a discrepancy function that quantifies the difference between experimentally measured and numerically computed data, through the test simulation. A damaged plasticity model is adopted as constitutive model within the simulation of the test. For a proper design of the procedure, first a sensitivity parametric analysis is performed to evaluate the influence of input parameters on the selected measurable quantities, namely the force vs. displacement curve of the wedge. The results evidenced that the curve is significantly influenced by all perturbed material parameters. Subsequently IA is performed to assess material parameters, specifically: Young's modulus, yield limit, and specific fracture energy. The IA procedure turned out to be robust and reliable to assess targeted parameters. The inverse problem developed is well-posed and the objective function is rather smooth as the procedure converges to the same values within multiple initializations within less than 10 iterations. The assessed values are compared against those obtained through more cumbersome procedure that employs additional equipment, namely a digital image correlation for full-field displacement measurements.

On solution to factorization problem for partly rational matrices of an arbitrary order

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The results of this report deals with the constructive factorization of matrix-functions (see, e.g., [1]), i.e. representation of the non-singular square $n \times n$ matrix-function G defined on a simple closed curve Γ in the following form

$$G(t) = G^{+}(t)\Lambda(t)G^{-}(t), \ t \in \Gamma, \tag{1}$$

where the matrices $G^{+}(t), G^{-}(t)$ possess an analytic continuation into domains $D^{+} = int\,\Gamma$, $D^{-} = ext\,\Gamma$, and $\Lambda(t)$ is a diagonal matrix

$$\Lambda(t) = diag\,\{t^{æ_1}, t^{æ_2}, \ldots, t^{æ_n}\}$$

with integer numbers $æ_1, æ_2, \ldots, æ_n$ (called partial indices of the factorization (1)).

The problem of factorization consists in determination of the factors $G^{+}(t), G^{-}(t)$ and calculation of partial indices $æ_1, æ_2, \ldots, æ_n$. It is an old problem dealing initially with the solution to vector-matrix boundary value problems, but now connecting with many mathematical questions and having a lot of applications (see [2]). Several methods were proposed to solve the factorization problem constructively including an approach by G.N.Chebotarev [3] who factorized triangular matrix-function applying the method of continued fraction.

In this work, it is proposed an algorithm for solution of the factorization problem for matrix-functions of arbitrary $n \times n$ order with partly rational entries. The algorithm is efficient for the case when $n-1$ columns or $n-1$ rows consist of rational functions (wlog the rational are the last $n-1$ columns). Our method generalizes the method of factorization of the triangular matrix function of an arbitrary order [4] and the method of reduction of the 2×2 matrix-function with a rational column/row to the triangular matrix-function [5].

Below we describe the scheme of our method. For simplicity we restrict our attention to the case of 3×3 matrix-functions with two rational columns defined and non-singular on the unit circle $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$:

$$G(t) = \begin{pmatrix} f_1(t) & Q_1(t) & P^{(m)}(t) \\ f_2(t) & Q_2(t) & P^{(n)}(t) \\ f_3(t) & Q_3(t) & P^{(k)}(t) \end{pmatrix}, \quad \det\, G(t) \neq 0,\, t \in \mathbb{T}. \tag{2}$$

Here $Q_1(t), Q_2(t), Q_3(t)$ are polynomials of certain (arbitrary) orders, $P^{(m)}(t), P^{(n)}(t), P^{(k)}(t)$ are polynomials of specified orders $k \le n \le m$ and $f_1(t), f_2(t), f_3(t)$ are arbitrary functions.

First step. Let us represent the ratio of polynomials $\frac{P^{(n)}(t)}{P^{(k)}(t)}$ in the (finite) continued fraction

$$\frac{P^{(n)}(t)}{P^{(k)}(t)} = S_0(t) + \cfrac{1}{S_1(t) + \cfrac{1}{S_2(t) + \cdots \cfrac{1}{S_l(t)}}}. \tag{3}$$

Then we multiply the matrix $G(t)$ from the left by the matrices $T_0, T_2, \ldots$

$$T_{2r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -S_{2r}(t) \\ 0 & 0 & 1 \end{pmatrix}, \quad r = 0, 2, \ldots, \tag{4}$$

and from the right by the the matrices $T_1, T_3, \ldots$

$$T_{2r+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -S_{2r+1}(t) & 1 \end{pmatrix}, \quad r = 0, 2, \ldots. \tag{5}$$

We obtain the matrix whose second element of the last column is equal to 0.

Second step. By multiplying this matrix by certain constant matrix we interchange the first and the second rows. We apply the same approach as before and to reduce the matrix to the following form

$$\begin{pmatrix} \tilde{f}_1(t) & \tilde{Q}_1(t) & 0 \\ \tilde{f}_2(t) & \tilde{Q}_2(t) & 0 \\ \tilde{f}_3(t) & \tilde{Q}_3(t) & P^{(\nu)}(t) \end{pmatrix}. \tag{6}$$

Third step. The next step is based on the following

Lemma. [4, Lemma 1] *Let Γ be a simple smooth closed contour, and $B(t), t \in \Gamma$ be a non-singular Hölder continuous square matrix-function of the order n having the following form:*

$$B(t) = \begin{pmatrix} A(t) & \mathbf{0} \\ b_1(t) \ldots b_{n-1}(t) & c(t) \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{7}$$

Suppose that the non-singular square matrix-function $A(t)$ of the order $n-1$ admits factorization

$$A(t) = X^{+}(t)\, \Lambda(t)\, X^{-}(t),$$

where $\Lambda(t) = \operatorname{diag}\{t^{\kappa_1}, \ldots, t^{\kappa_{n-1}}\}$. Then the matrix-function $B(t)$ possesses factorization if the following matrix does:

$$\begin{pmatrix} \Lambda(t) & \mathbf{0} \\ (\mathbf{b}(t)|\mathbf{Y}_1^{-}(t)) \ldots (\mathbf{b}(t)|\mathbf{Y}_{n-1}^{-}(t)) & c(t) \end{pmatrix}. \tag{8}$$

Here $\mathbf{b}(t) = (b_1(t), \ldots, b_{n-1}(t))$ is the row of first $n-1$ entries of the lowest row of $B(t)$, $\mathbf{Y}_j^{-}(t) = (y_{1j}^{-}(t), \ldots, y_{n-1,j}^{-}(t))^T$ is the j-th column of the matrix-function $Y^{-}(t) = (X^{-}(t))^{(-1)}$, and

$$(\mathbf{b}(t)|\mathbf{Y}_j^{-}(t)) = \sum_{k=1}^{n-1} b_k(t) y_{kj}^{-}(t).$$

Final step. Factorization of the matrix $A(t)$ corresponding to (6) with the second rational (polynomial) column is obtained by the method of [5].

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A directional localisation and a Willis-type coupling in two-scale homogenisation of generalised elastodynamic microresonances

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Macroscopic dynamic properties of composite materials containing "micro-resonant" inclusions can be very different from those of conventional materials. Mathematically this leads to studying homogenisation of problems with a "critically" scaled high contrast, where the resulting two-scale asymptotic behaviour appears to display a number of interesting effects. Reference [1] considers scenarios of such periodic elastic micro-resonances generally involving interconnectedness and high anisotropy, and derives an associated coupled two-scale limit system. For a more recent application to high-contrast periodic Timoshenko elastic beam lattice materials displaying band gap-type effects see [2].

In the talk, we discuss how the construction in [1] can generally lead to a coupled frequency and directional localisation, and to associated macroscopic elastodynamic constitutive relations displaying a Willis-type coupling.

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Recent advances in application of bi-orthogonality relations

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The reciprocity and bi-orthogonality relations are very useful but, unfortunately, still underestimated tools to solve a broad range of wave propagation problems in a surprisingly simple way. At first acquaintance, these relations promise to be a goldmine of useful information. It takes some ingenuity, however, to unearth the nuggets that are not immediately obvious from the formulation (J. Achenbach, Reciprocity in Elastodynamics, 2004). In this talk, we summarise the recent advances in formulation and application of bi-orthogonality relations. In the framework of analytical reduced order modelling, we extend their formulation to non-symmetric waveguides and promote their link with dispersion equations. We also demonstrate that these relations unambiguously define two sets of boundary conditions for conventional modal analysis of a slice of a waveguide of arbitrary complexity. In the case of a uniform waveguide, the dispersion diagram is easily recovered from the sets of eigenfrequencies. In the case of a periodic waveguide, modal analysis of a unit symmetric periodicity cell provides frequency-wise location of pass- and stop-bands.

Computational implementation and validation of constitutive models for heat-resistant devices

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In the past several decades, the reduction of the ecological footprint (including the harmful emissions of CO2 and the rest of the gases contributing to the greenhouse effect) has become a burning topic in the world, essential for the development of a sustainable environment. Due to the high performance of refractories in extreme conditions, such as thermal shock and working at high temperatures, they are massively used in steel production. We are witnessing an increasing demand for steel in various spheres of the populations everyday activities; therefore, the consumption of this material can be considered an absolute necessity. The mechanical properties of refractories are very different from those of metals, as they undergo pressure-sensitive yielding, dilatant or contractive plastic deformation, and frictional behavior. For this reason, knowledge about the behavior of these materials throughout the whole working temperature range, as well as obtaining tools for their optimized design, is fundamental. After defining the theoretical framework, the most recent IT technologies have the aim of achieving an efficient transition to user-friendly software applications. The developed software tools are further aimed at being directly employed for the design and optimization of heat-resistant devices. The constitutive model providing a solid starting point was developed by Massimo Penasa [1]. Modifications in the definition of the model were introduced in order to characterize the new user subroutine, employing features such as the Bigoni-Piccolroaz yield function [2] , a customized subroutine for the temperature-dependent elastic modulus, and the hardening law defined using Bezier curves. This function has been tested thoroughly with uniaxial and biaxial tests, as well as on the example of the heat-resistant device.

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Inverse problems of identifying the time-dependent source coefficient for subelliptic heat equations

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We discuss inverse problems of determining the time-dependent source coefficient for a general class of subelliptic heat equations. We show that a single data at an observation point guarantees the existence of a (smooth) solution pair for the inverse problem. Moreover, additional data at the observation point implies an explicit formula for the time-dependent source coefficient. We also discuss the case with nonlocal data. Our proofs are based on subelliptic spectral theory arguments and elements of the subelliptic potential theory. This talk is based on our recent work with Mansur Ismailov (Gebze Technical University) and Tohru Ozawa (Waseda University).

Algorithm for adaptive line insertion for anticipating crack path trajectory

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Brittle materials in general exhibit almost no plastic deformation prior to the onset of fracture, which is typically fast propagating with fairly small amount of energy dissipated during the cracking process. Typical representative of such materials are refractory materials. One of the challenges in simulating crack propagation is that within the stress field of deformed material, crack tip represents a singularity point with very high stress gradients. This represents a challenging modeling task as in principle, the path of crack further propagation depends to a large extent on the stress distribution around the crack tip. In order to model discontinuity in the displacement filed provoked by the crack, cohesive elements are popular choice within finite element framework. These elements are surfaceless (i.e. volumeless for 3D applications) and are used to simulate the crack tip as a cohesive zone, which is a region of material that is modeled through a traction-separation law. The traction-separation law defines the relationship between the traction and the separation (i.e. crack opening displacement) across the cohesive zone and thus helps in the description of the material behavior in the crack tip zone. The biggest drawback of employment of cohesive elements is that they have to be inserted within the mesh prior to the simulations, consequently making the crack path trajectory pre-defined by selected mesh. In this communication results with developed algorithm for the automatic insertion of cohesive elements will be presented. Based on this algorithm, in-house software is developed that uses adaptively inserted cohesive elements in the fracture zone to simulate crack initialization and propagation depending on the applied criterion. The algorithm has an option to model surrounding solid material with fairly complex constitutive models and can use different crack initiation and propagation criteria. Within the algorithm an iterative approach for element insertion is adopted to account for changes of stress distribution over the uncracked specimen as the crack propagates.

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Statistical mechanics informed continuum mechanics models for damage of biopolymers

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Biodegradable copolymers are successfully adopted in numerous biomedical applications thanks to their biocompatibility and biodegradability. Their mechanical response under cyclic loading is also fundamental in the everyday medical practice and so is the prediction of residual stretches, toughness, stress softening, good tensile and knot-pull strengths.

In this contribution, we use a fully three-dimensional point-wise distribution of polymeric chains whose properties vary with stretch history. As well known, soft materials are characterized by a fundamental anisotropy of damage since each chain natural configuration depends on its own extension history. Specifically, in the framework of statistical mechanics, we propose a constitutive approach in which the Helmholtz free energy stems from two terms: an entropic Worm Like Chain (WLC) component and a network topological one [1, 2]. The total energy, again based on statistical mechanics results, evolve both in natural and contour lengths for each chain. In particular we describe the copolymer molecules as a two phase material with hard (folded) and soft (amorphous) domains. Due to externally imposed stretching, folded domains undergo a hard→soft transition with an increase of available monomers producing a contemporary growth of the contour and natural lengths. By following the approach proposed in [3] we the second energy term takes care of the network topological constraint. Based on the deduced filament Helmholtz free energy, we consider an initial isotropic chains distribution of the copolymer [4,5]. As a result, for an assigned macro deformation history, we are able to deduce the chain stretch in each direction that, after numerical integration, gives back the total internal specific energy with history dependent parameters, describing anisotropic damage. This novel macroscopic model is characterized by a set of five physics informed parameters only: one for each component of the free energy, one relating the contour length to the natural end-to-end chain length and two describing refolding and unfolding in unloading and reloading. In this way we are able to capture both anisotropic damage, residual stretches, and hysteresis. Our model is validated with experimental data on biodegradable copolymer suture threads and numerically explored for more general scenarios.

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Author Index