Numerical modeling of hydraulic fractures: State of art and new results

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Abstract

Hydraulic fracturing is one of the most efficient methods to increase production of oil, gas and heat from underground reservoirs. Its numerical modeling has been the subject of numerous publications. The paper briefly summarizes their results and presents recent findings, which notably improve numerical modeling. The conclusions are drawn on new options and further work for enhancing numerical modeling of hydraulic fractures.

1 Introduction

Hydraulic fracturing is a technique used extensively to increase the surface to or from which a fluid flows in a rock mass. Beginning with the papers [1], [2], [3], [4], [5], [6], [7], numerous studies have been published on the theory and numerical simulation of hydraulic fracturing (see, e. g., the papers [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] and detailed reviews in many of them). They have provided knowledge on the asymptotics of the solution, possibility to neglect the lag between the liquid front and the fracture contour and on the typical regimes. The knowledge was incorporated in the computational codes for practical applications (e.g. [11], [14]). Still, there is the need "to dramatically speed up" simulators [14].

The goal cannot be reached without clear understanding of underlying computational difficulties which strongly influence the accuracy and stability of numerical results and robustness of procedures. The recent studies of the author [18], [19], [20], [21], tended to address this challenge, have disclosed hidden features of the problem important for numerical modeling. They have led to the modified formulation of the problem, which opens new options for improving simulators.

The paper aims to (i) clearly explain the conventional formulation, (ii) present the recent findings, summarized in the modified formulation, (iii) demonstrate the advantages of the latter, and (iv) make conclusions on the further work.

2 Conventional formulation

A mathematical formulation of the problem includes (i) fluid, (ii) solid, and (iii) fracture mechanics equations. Their conventional forms are as follows.

Fluid equations. They include the equation of the mass conservation and the Poiseuille type eq1uation for flow in a narrow channel. For incompressible fluid, the mass conservation means the volume conservation:

$$\frac{\partial w}{\partial t} + \mathrm{div}\mathbf{q} + q_l = 0, \tag{1}$$

where $w(\mathbf{x}, t)$ is the channel width (fracture opening), $\mathbf{q}(\mathbf{x}, t)$ is the flux vector through the fracture height, $q_l(\mathbf{x}, t)$ is the intensity of distributed sources (usually this term accounts for leak-off and assumed positive), $\mathbf{x}$ denotes the vector of the position of a point on the surface of the flow, t is the time. The flux and divergence are defined in the tangent plane to the surface of the flow.

The Poiseuille type equation is of the form

$$\mathbf{q} = -D(w, p)\mathrm{grad}p, \tag{2}$$

where $p(\mathbf{x}, t)$ is the net-pressure, D is a function or operator, such that $D(0, p)\mathrm{grad}p = 0$. Gradient is also defined in the tangent plane.

Substitution of (2) into (1) yields the lubrication (Reynolds) equation:

$$\frac{\partial w}{\partial t} - \mathrm{div}(D(w, p)\mathrm{grad}p) + q_l = 0. \tag{3}$$

An initial spatial distribution $w_0(\mathbf{x})$ of the opening is defined at start time t_0:

$$w(\mathbf{x}, t_0) = w_0(\mathbf{x}). \tag{4}$$

The spatial operator in (3), being elliptic of the second order, it requires only one boundary condition (BC) at the fluid contour L_f. When neglecting the lag between the fluid front L_f and the fracture contour L_c, it may be the condition of the prescribed normal component q_n of the flux:

$$q_n(\mathbf{x}) = q_0(\mathbf{x}), \quad \mathbf{x} \in L_f, \tag{5}$$

where $q_0(\mathbf{x})$ is a known function at L_f; at the points of the fluid injection it is defined by the injection regime; at the points of fluid front, coinciding with the fracture contour, we have $w = 0$ and equation (2) implies $q_0(\mathbf{x}) = 0$.

Solid mechanics equations define a dependence of the opening on the net-pressure caused by deformation of rock:

$$Aw = p, \tag{6}$$

with the condition of zero opening at points of the fracture contour $\mathbf{x}_c$:

$$w(\mathbf{x}_c) = 0. \tag{7}$$

Commonly, the operator A in (6) is obtained by using the theory of linear elasticity. As mentioned, when neglecting the lag, the condition of zero opening (7) replaces the condition of zero flux on the front. Henceforth, we shall consider this case and write $\mathbf{x}_c = \mathbf{x}_*$ with the star marking that a quantity refers to the fluid front.

Fracture mechanics equations define the critical state and the perspective direction of the fracture propagation. In the commonly considered case of the tensile mode of fracture, these are:

$$K_I(\mathbf{x}_c) = K_{IC}, \qquad K_{II}(\mathbf{x}_c) = 0, \tag{8}$$

where K_I is the tensile stress intensity factor (SIF), K_{IC} is its critical value, K_{II} is the shear SIF.

The problem consists in solving the PDE (3) together with the elasticity equation (6) under the initial condition (4), boundary conditions (5), (7) and the fracture conditions (8). The global mass balance is usually employed to follow the fluid front propagation (e.g. [7], [9], [11], [14]).

3 Fluid particle velocity. Speed equation

The equations of the conventional formulation do not contain the average velocity of fluid particles in a narrow channel. Rather it employs the flux $\mathbf{q}$. The latter, by definition, is the particle velocity averaged across the opening and multiplied by the opening

$$\mathbf{q}(\mathbf{x}) = \mathbf{v}(\mathbf{x})w(\mathbf{x}). \tag{9}$$

Meanwhile, the particle velocity is the primary quantity used when deriving the mass conservation equation and the Poiseuille type equation. It and its averaged (across opening) value are significant from the physical and computational points of view. Of special importance is that the limit value of the average particle velocity at the fluid front v_{n*} represents the speed of the front propagation V_* [18]:

$$V_* = \frac{dx_{n*}}{dt} = v_{n*}(\mathbf{x}_*). \tag{10}$$

Herein, x_{n*} is the normal component of point $\mathbf{x}_*$ on the front. It is assumed that sucking or evaporation through the front is negligible.

In view of (9), the speed equation (10) may be written as

$$V_* = \frac{q_{n*}}{w_*}. \tag{11}$$

For the flux, defined by the Poiseuille type dependence (2), it specifies the *speed equation (SE) for a flow of incompressible fluid in a narrow channel* [18]-[21]:

$$V_* = \frac{dx_{n*}}{dt} = -\frac{1}{w_*(x_*)} D(w,p) \frac{\partial p}{\partial n}_{x=x_*}. \tag{12}$$

Thus we have the *local* condition (12) at points of the propagating fluid front. This allows one to trace the propagation by well-developed methods of the theory of propagating interfaces (see, e. g. [22]). In contrast, the conventional formulation employs the *global* mass balance (e.g. [7], [9], [11], [14]), which is a *single* equation. The latter is sufficient when considering 1-D problems with one point of the front to be traced. However, in the general case of 2D fracture, it is preferable to employ the SE, which is formulated at *each* of many traced points of the fluid front. This gives the first evidence that using the particle velocity is beneficial from the computational point of view.

The next evidence follows from the definition of the flux (9). In view of (2) it implies:

$$\mathbf{v} = \frac{\mathbf{q}}{w} = -\frac{1}{w}\mathsf{D}(w,p)\,\mathrm{grad}p. \tag{13}$$

From (10) and (13) we see that even when $w_*(x_*) = 0$ and $q_{n*} = 0$, the limit of the ratio $\mathbf{q}/w$ should be finite to exclude the front propagation with infinite velocity. Thus near the front, where both the flux and the opening rapidly decrease, their ratio, representing the particle velocity, does not change thus fast being finite and non-zero. Moreover, the particle velocity is non-zero in the entire flow region except for flows with stagnation points. From (13), it can be also seen that the particle velocity is notably smoother function than the pressure. Therefore, the particle velocity is a better choice as an unknown function in the lubrication equation than the flux or the pressure. We conclude that *it is reasonable to employ the particle velocity* for numerical modeling of hydraulic fracture propagation. Below it will be shown that the equality of the particle velocity at the front to the propagation speed (10), provides additional computational advantages.

4 Clear evidence that BVP is ill-posed

4.1 Nordgren problem

The SE (12) is additional to a prescribed boundary condition at the points of the fluid front. For zero lag, this leads to difficulties common to over-determined problems when solving the boundary value problem (BVP) numerically for a fixed position of the front on an iteration. To disclose the difficulties and to find a means to overcome them, we study the Nordgren problem [6].

The Nordgren model considers a straight fracture of the height h propagating along the x-axis under plain-strain conditions. Then the net-pressure in equation (6) is proportional to the opening: $p = k_e w$, where $k_e = (2/\pi h)E/(1-\nu^2)$, E is the Young's modulus, ν is the Poisson's ratio of rock mass. The fluid is assumed Newtonian and consequently the operator D in (2) is the multiplier $\mathsf{D}(w,p) = k_l w^3$, where in the case of an elliptic cross section considered by Nordgren $k_l = 1/(\pi^2\mu)$, μ is the dynamic viscosity.

For simplicity, we neglect leak-off and use the dimensionless variables: $x_d = x/x_n$, $x_{*d} = x_*/x_n$, $w_d = w/w_n$, $y_d = y/y_n$, $v_d = v/v_n$, $p_d = p/p_n$, $q_d = q/q_n$, $q_{0d} = q_0/q_n$, where $x_n = (k_l k_e/4)^{1/5} q_n^{3/5} t_n^{4/5}$, $w_n = q_n t_n/x_n$, $y_n = w_n^3$, $v_n = x_n/t_n$,

$p_n = k_r w_n/4$, q_n, and t_n are *normalizing* length, opening, cubed opening, particle velocity, flux and time, respectively. The normalizing quantities q_n, t_n may be chosen as convenient. From this point on, we omit the subscript d at the normalized variables and consider only dimensionless values. The PDE (3) becomes

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w^4}{\partial x^2} = 0. \tag{14}$$

The initial condition (4) in the 1-D case reads

$$w(x, t_0) = w_0(x), \tag{15}$$

with $w_0(x) = 0$ ahead of the fluid front x_*. The BC (5) of the prescribed influx q_0 at the inlet $x = 0$ and the BC (7) of zero opening at the front $x = x_*$ are, respectively,

$$-\frac{\partial w^4}{\partial x}_{x=0} = q_0, \tag{16}$$

$$w(x_*, t) = 0. \tag{17}$$

The SE (12) is not used in the conventional formulation. In the dimensionless variables it reads:

$$V_* = \frac{dx_*}{dt} = -\frac{4}{3}\frac{\partial w^3}{\partial x}_{x=x_*(t)}. \tag{18}$$

We see that the PDE (14) is of *second order* in the spatial variable x, while there are *three* rather than two *boundary conditions* (16)-(18) for any fixed position of the front x_*. It can be shown (see subsection (4.3)) that under the BC (17), in limit $x \to x_*$, the PDE (14) turns into the SE (18). Thus one may expect difficulties when trying to solve the problem (14)-(17) numerically under fixed x_* at each iteration within a time step. Further discussion confirms this suggestion.

4.2 Straightforward solving BVP for starting PDE

Nordgren [6] used straightforward numerical integration of the problem (14)-(17). This author applied Crank-Nicolson finite difference scheme to approximate the PDE (14) and to meet the BC (16), (17). The SE (18) was not mentioned. The paper [6] does not contain details of calculations on the initialization, the time step, the number of nodes in spatial discretization, the number of iterations, stability of numerical results and expected accuracy. To obtain knowledge on these issues, we also solved the BVP (14)-(17) in a straightforward way by using the Crank-Nicolson scheme. The results are as follows [19].

Actually performing 20 iterations to account for the non-linear term w^4 is sufficient to reproduce four digits of the fracture opening, except for a close vicinity of the liquid front. (Increasing the number to 100 iterations does not improve the solution for all tested time and spatial steps.). For various time steps ($\Delta t = 10^{-2}$, 10^{-3}, 10^{-4}) and different spatial steps (10^{-2}, 10^{-3}, 10^{-4}) taken in various combinations, the results are stable along the main part of the interval $[0, x_*(t)]$. However, *the results always deteriorate and they are unreliable in a close vicinity of the front*

$(1 - x/x_* < 0.001)$. The results coincide with those given in the paper [6] to the accuracy of two significant digits accepted in this work. In all the calculations, *by no means could we have a reliable third digit* not only near the front but in the entire flow region. Fine spatial meshes did not improve the accuracy as compared with a rough mesh having the step 0.01. Moreover, using very fine spatial meshes with the step less than 10^{-5} led to complete deterioration of the solution in the entire flow region.

The numerical results clearly show that the BVP with a fixed position of the front at an iteration cannot be solved accurately without regularization. The *problem appears ill-posed* in the Hadamard sense [23].

4.3 Straightforward solving BVP for ODE of self-similar formulation

To further clarify the essence of the difficulties, we employ the fact that the Nordgren problem does not include characteristic geometrical and time parameters. Consequently, its self-similar formulation becomes available [6], [7]. For the constant influx q_0, the self-similar variables are: the self-similar coordinate $\xi = xt^{-4/5}$, the self-similar opening $\psi(\xi)$, the self-similar particle velocity $v_\psi(\xi) = -\frac{4}{3}\frac{d\psi^3}{d\xi}$ and the self-similar fracture length $\xi_* = x_* t^{-4/5}$. They define the physical quantities as the functions with separated temporal t^β and spatial $\sigma = x/x_* = \xi/\xi_*$ variables: $w(t,x) = t^{1/5}\psi(\xi)$, $v(t,x) = t^{-1/5}v_\psi(\xi)$, $x_* = \xi_* t^{4/5}$. The self-similar front speed is $V_{\psi *} = 0.8\xi_*$; the physical speed is $V_* = \frac{dx_*}{dt} = 0.8\xi_* t^{-1/5}$.

In terms of the self similar quantities, the PDE (14) becomes the ODE:

$$y\frac{d^2y}{d\xi^2} + \frac{1}{3}\left(\frac{dy}{d\xi} + 0.6\xi\right)\frac{dy}{d\xi} - \frac{3}{20}y = 0, \tag{19}$$

where $y(\xi) = \psi^3(\xi)$. The BC (16) and (17) become, respectively,

$$\sqrt[3]{y(0)}\frac{dy}{d\xi}_{\xi=0} = -\frac{3}{4}q_0, \tag{20}$$

$$y(\xi_*) = 0. \tag{21}$$

The SE (18) in the self-similar form reads:

$$\frac{dy}{d\xi}_{\xi=\xi_*} = -0.6\xi_*. \tag{22}$$

In limit $\xi \to \xi_*$, for a solution, satisfying the BC (21), the ODE (19) turns into the SE (22). Hence, for the ODE (19), at the point ξ_*, we have imposed not only the BC (21) for unknown function y, but also the BC (22) for its derivative $dy/d\xi$. The problem appears ill-posed. The following discussion makes it obvious.

Re-write the ODE (19) as

$$\frac{d^2y}{d\xi^2} + a(y, dy/d\xi, \xi)\frac{dy}{d\xi} - \frac{3}{20} = 0, \tag{23}$$

where $a(y, dy/d\xi, \xi) = (dy/d\xi + 0.6\xi)/(3y)$. The equations (21), (22) imply that the factor a in (23) is finite at the fluid front.

It is easy to check by direct substitution that if $y_1(\xi_1)$ is the solution of the problem (19)-(21) for $q_0 = q_{01}$ with $\xi_* = \xi_{*1}$, then $y_2(\xi_2) = y_1(\xi_2\sqrt{k})/k$ is the solution of the problem ()-() for $q_{02} = k^{-5/6}q_{01}$ with $\xi_{*2} = \xi_{*1}/\sqrt{k}$; herein, k is an arbitrary positive number. This implies that $C_* = q_0^{0.6}/\xi_*$ and $C_0 = y(0)/\xi_*^2$ are constants independent on the prescribed influx q_0. As $\xi_* = q_0^{0.6}/C_*$, it is a matter of convenience to prescribe q_0 or ξ_*. A particular value of q_0 or ξ_* may be also taken as convenient. Indeed, with the solution $y_1(\xi_1)$ for $q_0 = q_{01}$, we find the solution for any q_0: $y(\xi) = y_1(\xi\sqrt{k})/k$, where $k = (q_{01}/q_0)^{6/5}$, $\xi = \xi_1/\sqrt{k}$ ($\xi_* = \xi_{*1}/\sqrt{k}$).

Let us fix ξ_*. According to (21), (22), at the point ξ_*, we have prescribed both the function y and its derivative $dy/d\xi$. Thus, for the ODE of the second order (19) we have a Cauchy (initial value) problem. Its solution defines $y(0)$, $\frac{dy}{d\xi}_{\xi=0}$ and consequently the flux q_0 at $\xi = 0$. Hence, even a small error when prescribing q_0 in (20), excludes the existence of the solution of the BV problem (19)-(21). Therefore, by Hadamard definition [23], the BV problem (19)-(21) is ill-posed. It cannot be solved without a proper regularization [24].

Direct computations confirm that it is impossible to accurately solve the BVP (23), (20), (21). We performed hundreds of numerical experiments with various numbers of nodal points and iterations and different values of the prescribed influx q_{01} at the inlet. Finite difference approximations of second order for $d^2y/d\xi^2$ and $dy/d\xi$ were combined with iterations for the non-linear term $a(y, dy/d\xi, \xi)$. Up to 100 000 nodal points and up to 1500 iterations were used in attempts to reach the accuracy of three correct digits, at least. The attempts failed: by no means could we have more than two correct digits in the entire flow region. Moreover, the results always strongly deteriorate near the fluid front. The numerical results clearly demonstrate that the BV problem (19)-(21) is ill-posed. It cannot be solved accurately without regularization.

5 e-regularization

5.1 e-regularization for self-similar formulation

A regularization method is suggested by the conditions (21), (22). Indeed, we may use them together to get the approximate equation $y \approx 0.6\xi_*(\xi_* - \xi)$ near the front. Hence, instead of prescribing the BC (21) at the fluid front $\xi = \xi_*$, where it is implicitly complemented by the SE (22), we may impose the boundary condition, which combines (21) and (22) at a point $\xi_\varepsilon = \xi_*(1 - \varepsilon)$ at a small relative distance $\varepsilon = 1 - \xi_\varepsilon/\xi_*$ from the front:

$$y(\xi_\varepsilon) = 0.6\xi_*^2(1 - \varepsilon). \tag{24}$$

The BV problem (19), (20), (24) is well-posed and it may be solved by finite differences. Numerical implementation of this approach shows that with $\varepsilon = 10^{-3}$, 10^{-4}, the results for the step $\Delta\zeta = \Delta\xi/\xi_* = 10^{-3}$, 10^{-4}, 10^{-5}, 10^{-6} coincide with those of the benchmark solution [19]. The time expense is fractions of a second. The results

are stable if ε and $\Delta\zeta$ are not simultaneously too small (both ε and $\Delta\zeta$ are greater than 10^{-5}).

The essence of the suggested regularization consists in using the SE together with a prescribed BC to formulate a BC at a small distance behind the liquid front rather than on the front itself. Besides, the SE is also imposed at the point ξ_ε; it is used for iterations. We call such an approach e-regularization.

5.2 e-regularization for starting PDE

Extension of e-regularization to solve the starting PDE (14) requires using the BC (17) on the front combined with the SE (18) to impose a BC at a small relative distance from the front. Introduce the relative distance $\eta = (x_* - x))/x_*$ from the front. The relative distance from the inlet is $\sigma = 1 - \eta = x/x_*$. When using the variable σ, the PDE (14) becomes:

$$\frac{d^2Y}{d\xi^2} + A(Y, \partial Y/\partial\sigma, \sigma)\frac{dY}{d\sigma} - B(Y, x_*)\frac{\partial Y}{\partial t} = 0, \tag{25}$$

where $Y(\sigma, t) = w^3(\sigma x_*(t), t)$, $A(Y, dY/d\sigma, \sigma) = (\partial Y/\partial\sigma + 0.75 x_* V_* \sigma)/(3Y)$, $B(Y, x_*) = x_*^2/(4Y)$. The BC (16), (17) in the new variables read:

$$\frac{\sqrt[3]{Y(0)}}{x_*}\frac{\partial Y}{\partial\sigma}_{\sigma=0} = -\frac{3}{4}q_0, \tag{26}$$

$$Y(\sigma, t)_{\sigma=1} = 0. \tag{27}$$

The SE (18) takes the form:

$$\frac{\partial Y}{\partial\sigma}_{\sigma=1} = -0.75 x_* V_*. \tag{28}$$

Note that in view of the conditions (27) and (28), the factor $A(Y, dY/d\sigma, \sigma)$ and the term $B(Y, x_*)\partial Y/\partial t$ are finite at the fluid front $\sigma = 1$. Note also that under the BC (27), in limit $\sigma \to 1$, the PDE (25) turns into the SE (28). In terms of the starting problem, this means that under the BC (17), in limit $x \to x_*$, the PDE (14) turns into the SE (18).

The regularization of the problem (25)-(27) follows the line used for the self-similar formulation. The BC (27) is combined with the SE (28) to obtain the approximate equation near the liquid front:

$$Y(\sigma, t) \approx 0.75 x_*(t) V_*(t)(1 - \sigma). \tag{29}$$

Thus we may impose the BC at a small relative distance ε from the front by taking equality sign in (29):

$$Y(\sigma_\varepsilon, t) = 0.75 x_*(t) V_*(t)\varepsilon, \tag{30}$$

where $\sigma_\varepsilon = 1 - \varepsilon$. In contrast with the problem (25)-(27), the problem (25), (26), (30) does not involve an additional BC. We may expect that it is well-posed and provides the needed regularization. Extensive numerical tests confirm the expectation [19].

We solved the problem (25), (26), (30) by using the Crank-Nicolson scheme and iterations for non-linear multipliers $A(Y, dY/d\sigma, \sigma)$ and $B(Y, x_*)$ at a time step. The velocity was also iterated by imposing the SE at the point $\sigma_\varepsilon = 1 - \varepsilon$. We could see that with $\varepsilon = 0.0001$, $\Delta\sigma = 0.01$, and fifty iterations of the non-linear terms at a time step, the relative error of the fracture length x_* and the front speed x_* was less than 0.03% at each of 20000 time steps. Moreover, starting from the relatively small time $t = 0.01$, we could reach the time $t = 36128$ without loss of accuracy. There were no signs of instability or deterioration of the opening near the front in these and many other specially designed experiments. Therefore, e-regularization is quite efficient.

5.3 e-regularization in general case

According to the rationale presented in the preceding subsections, it appears that the strategy of using e-regularization when tracing 2-D hydrofracture propagation is as follows. At each point of the liquid front, an exact boundary condition is changed to an approximate equality at a small distance r_ε behind the front. This approximate equality is obtained by combining the boundary condition at the fluid front, particular for a considered problem, with the SE, which is quite general. In practical calculations, the distance (absolute r_ε or relative ε) is taken small enough to use the equality sign in the derived approximate condition. This gives us the *e-regularized boundary condition* near the front. The SE is also assumed to be met at the distance r_ε with an accepted accuracy. This gives us the *e-regularized speed equation.* The e-regularized boundary condition allows one to avoid unfavorable computational effects; the e-regularized SE serves to find the front propagation.

In this way, in general, the speed equation (12) is combined with a condition on the fluid front to obtain the e-regularized boundary condition [19], [20]:

$$\int_{p_*}^{p_\varepsilon} \frac{1}{w} D(w, p)\, dp = V_* r_\varepsilon, \tag{31}$$

where p_* is the pressure at the front, $p_\varepsilon = p(r_\varepsilon)$ is the pressure at the distance r_ε from the front.

The e-regularized form of the SE (12) is:

$$V_*(t) = \frac{dx_{n*}}{dt} = -\frac{1}{w} D(w, p) \frac{\partial p}{\partial n}_{r_\varepsilon}. \tag{32}$$

For the Nordgren problem, in the normalized variableswe have $p = 4w$, $p_* = 4w(x_*) = 0$, $D(w, p) = w^3$, $r_\varepsilon = \varepsilon x_*$; then, since $Y = w^3$ and $\partial/\partial n = -\frac{1}{x_*}\partial/\partial\sigma$, the regularized BC (31) and the regularized SE (32) reduce to (30) and $\partial Y/\partial\sigma_{\sigma=\sigma_\varepsilon} = -0.75 x_* V_*$, respectively.

6 Choice of proper variables

In Section 2, it has been shown that the *particle velocity* is a better choice as an unknown function in the lubrication equation than the flux or the pressure. A proper

choice of the *spatial variables* near the liquid front is suggested by e-regularization and by the asymptotic behavior of the opening near the liquid front. Recall that the e-regularized equations (31) and (32) actually employ the system moving with the front. Thus it is reasonable to re-write the lubrication equation (3) in this system. In it, the r-axis is directed opposite to the front velocity, while the other axis is tangent to the front. Then equation (3) becomes [19]:

$$\frac{\partial \ln w}{\partial t} = \frac{\partial v_n}{\partial r} + (v_n - V_*)\frac{\partial \ln w}{\partial r} - \frac{1}{w} q_l, \tag{33}$$

where using $\ln w$ serves to account for an arbitrary power asymptotic behavior of the opening $w(r,t) = C(t)r^\alpha + O(r^\delta)$ near the front ($\alpha \geq 0$, $\delta > \alpha$). The value of the exponent α is known in a number of important particular cases, $\delta = 1 + \alpha$ when the leak off is neglected (see, e.g. [7], [8], [9], [15]).

When the opening has the power asymptotic near the front with $\alpha > 0$, it yields singular behavior of the spatial derivatives $\partial w/\partial r$, $\partial^2 w/\partial r^2$ at the front, and this complicates numerical solution of a problem. Therefore, *it is reasonable, in addition to the particle velocity, to use the variable* $Y = w^{1/\alpha}$, which is linear near the front. In terms of the variables Y, v_n and r, the lubrication equation (3) near the liquid front becomes

$$\frac{\partial Y}{\partial t} = \frac{Y}{\alpha}\frac{\partial v_n}{\partial r} + (v_n - V_*)\frac{\partial Y}{\partial r} - \frac{Y^{1-\alpha}}{\alpha} q_l. \tag{34}$$

The initial condition (4), boundary condition (5) and the SE (12) are easily re-written in these variables.

In 1-D cases, the PDE (34) is applicable to the entire fluid. In these cases, there is the only spatial coordinate x and it is reasonable to normalize x or, what is actually equivalent, r by the distance $x_*(t)$ from the inlet to the front. Then in terms of $\sigma = x/x_* = 1 - r/x_*$, the lubrication equation (34) in 1-D cases reads:

$$\frac{\partial Y}{\partial t} = \frac{1}{x_*}\left[(\sigma V_* - v)\frac{\partial Y}{\partial \sigma} - \frac{Y}{\alpha}\frac{\partial v}{\partial \sigma}\right] - \frac{Y^{1-\alpha}}{\alpha} q_l, \tag{35}$$

where we have omitted the subscript n in the notation of the particle velocity. Note that when q_l near the front decreases faster than $w = Y^\alpha$, we may divide (35) by Y, obtaining the equation

$$\frac{1}{Y}\frac{\partial Y}{\partial t} = \frac{\sigma V_* - v}{x_* Y}\frac{\partial Y}{\partial \sigma} - \frac{1}{\alpha x_*}\frac{\partial v}{\partial \sigma} - \frac{1}{\alpha Y^\alpha} q_l, \tag{36}$$

where under the assumed asymptotics of q_l, the term $(\partial Y/\partial t)/Y$, the factor $(\sigma V_* - v)/(x_* Y)$ and the derivative $\partial v/\partial \sigma$ are finite, while the term $q_l/(\alpha Y^\alpha)$ tends to zero at the liquid front.

7 Modified formulation. Computational and analytical advantages

7.1 Modified formulation

Employing the suggested variables and e-regularization results in the modified formulation of the hydraulic fracture problem. In contrast with the conventional for-

mulation it uses [18]-[21]:

1. the particle velocity, as a variable smooth near the liquid front, instead of the pressure;

2. the opening taken in a degree, defined by its asymptotic behavior at the liquid front, instead of the opening itself;

3. the speed equation at each point of the front to trace the fracture propagation by the well-developed methods [22], instead of the commonly employed single equation of the global mass balance; the speed equation also presents the basis for proper regularization;

4. e-regularization, that is imposing the boundary condition and the speed equation at a small distance from the front rather than on the front itself, to exclude deterioration of the solution near the front caused by the fact [18],[19] that the BVP is ill-posed for a fixed position of the front when neglecting the lag;

5. the spatial coordinates moving with the front and evaluation of the temporal derivative under fixed values of these coordinates;

6. reformulation of the common system of equations and boundary conditions in terms of the suggested variables complemented, when appropriate, with e-regularization.

7.2 Advantages of modified formulation

The *computational advantages* of using the modified formulation have been explained in the course of the exposition. There are also *analytical advantages*, which appear due to smoothness of the new variables (particle velocity and $Y = w^{1/\alpha}$) near the fluid front. In 1-D cases, they allow one to obtain analytical solutions of problems, like those by Nordgren [6], Spence & Sharp [7], which otherwise require involved calculations. The analytical solutions of these problems may be found in the paper [21]. Further obvious applications of the analytical approach may include accounting for leak-off, when the latter is prescribed in separated temporal and spatial variables with a specially chosen temporal part and with the spatial part having the same asymptotic near the fluid front as the opening. Analogous axisymmetric problems may be solved in this way, as well.

8 Conclusions on further work

Further work on enhancing numerical modeling of hydraulic fractures may employ new facilities suggested by the modified formulation of the problem. Some of them have been mentioned above.

1. Since the SE is formulated at each point of the front, it notably extends options for tracing the fracture propagation as compared with the traditional approach employing the single equation of the global mass balance. The SE opens

the possibility to use the well-developed numerical methods of the theory of propagating interfaces [22]. In particular, level set methods and fast marching methods become of use.

2. New efficient iterative schemes may employ the particle velocity as an unknown function, which is notably smoother than commonly employed net-pressure. What also looks beneficial, only the *first* spatial derivatives of the particle velocity enter the modified lubrication equation. The same refers to using the opening at the degree defined by its asymptotic behavior near the front.

3. Employing e-regularization provides an opportunity for examining and improving the accuracy of existing commercial codes serving for modeling hydraulic fractures.

4. For an area sufficiently close to the front, where the solution changes faster than at the remaining part of a fracture, the PDE (34) and the SE (12), after spatial discretization, suggest efficient integration in time of a non-linear system of ODE under initial (Cauchy) conditions by using multi-stage methods like the Runge-Kutta methods.

5. Obtaining analytical solutions accounting for leak-off in 1-D plain-strain and axisymmetric problems. Using these solutions for accurate description of the boundary layer effects caused by the existence of the lag between the fluid front and the fracture contour.

6. Working out improved while simple models and numerical schemes for accounting for the proppant movement.

Perhaps, execution of these works may facilitate progress in solving even more difficult and important problems of hydraulic fracturing concerning with strong inhomogenuity of rocks and presence of multiple contacts and natural cracks, which may serve as channels for fluid flow. Still the greatest challenge is to comprehend and to properly model hydraulic fractures in low permiable shales.

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