Speed Equation in Problems of Hydraulic Fracturing:
Theory and Applications

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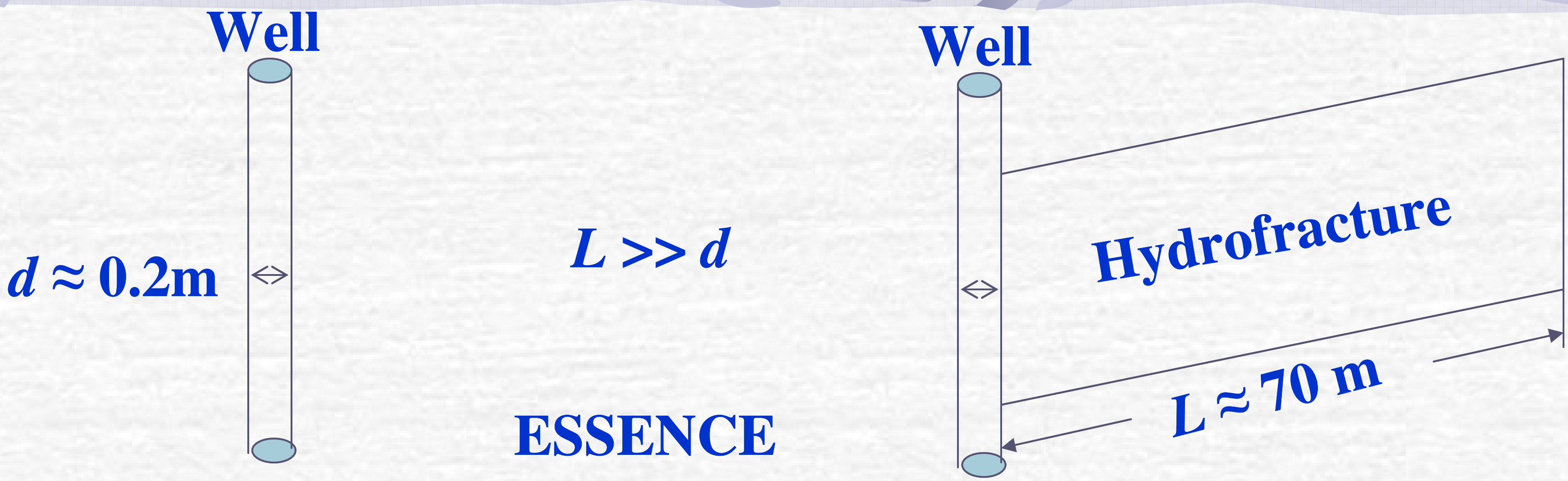
Hydraulic Fracturing

Essence and Brief Historical Overview



ESSENCE

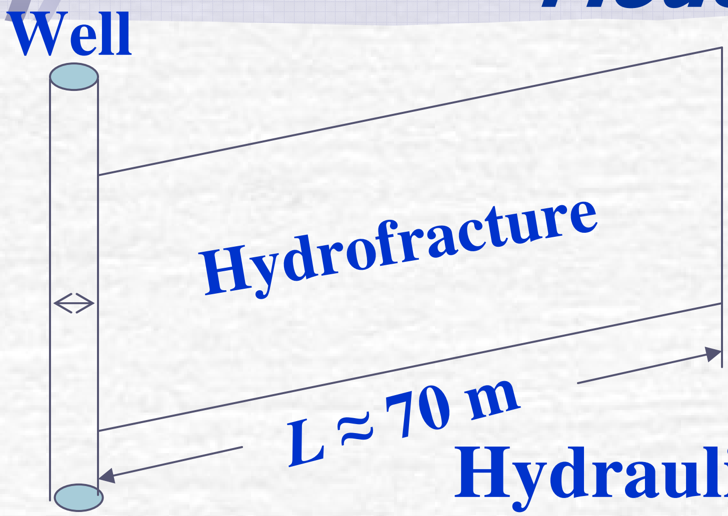
Drastic increase of the surface to which oil flows to the well

HISTORIC OVERVIEW

1896 USA Patent No 556 669 pumping fluid under pressure to force acid further into rock; 1930s Dow Chemical Company discovered that fluid pressure could be applied to crack and deform rock leading to better well stimulation; 1947 First hydraulic treatment to stimulate well production in order to compare with the current technology (Kansas, Hugoton field)

Hydraulic Fracturing

Modern Applications



Today, hydraulic fracturing is used extensively in the petroleum industry to stimulate oil and gas wells in order to increase their productivity.

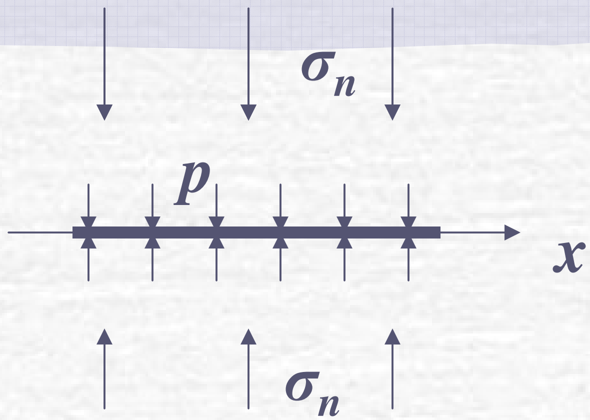
Hydraulic fracturing is also used to

- **Increase heat production from geothermal reservoirs**
- **Measure in-situ stresses**
- **Control caving of roof in coal and ore excavations**
- **Enhance CO_2 sequestration**
- **Isolate toxic substances in rock**

Thousands of treatments are successfully pumped each year

In natural conditions **pressurized melted substance fractures earth crust leading to formation of veins of mineral deposits**

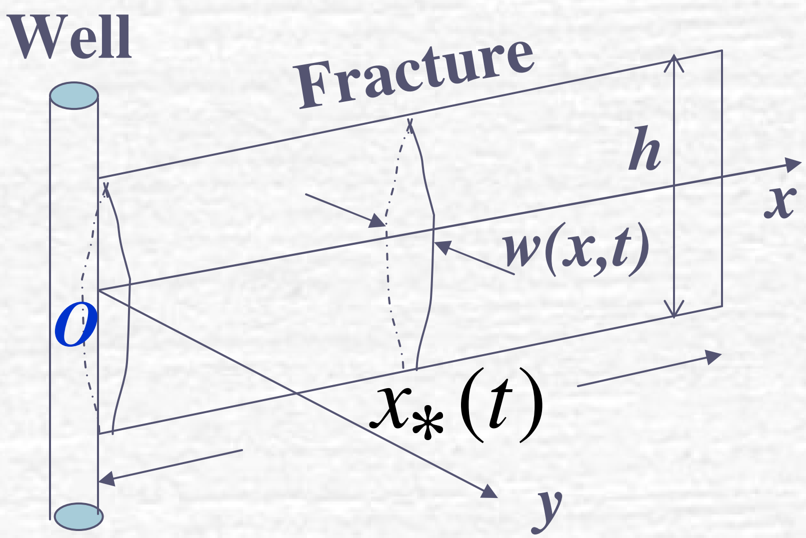
First Theoretical Models



KGD model; horizontal cross section

Khristianovich & Zheltov 1955

Geertsma & de Klerk 1969



PKN model; vertical cross section

Perkins & Kern 1961

Nordgren 1972

Howard & Fast (1970) Hydraulic Fracturing Monograph Series Soc. Petrol. Eng.

Further Theoretical Work

Studying of asymptotics and self-similar solutions

Numerous papers on theoretical studying of hydraulic are focused on

(i) asymptotics at crack tip;

(ii) self-similar and asymptotic solutions to study regimes of flow

Spence & Sharp 1985: **self-similar plane problem and asymptotics for newtonian liquid;**

Desrouches, Detournay et al 1994: **asymptotics for power-law liquid;**

Adachi & Detournay 2002: **self-similar plane problem for power-law liquid;**

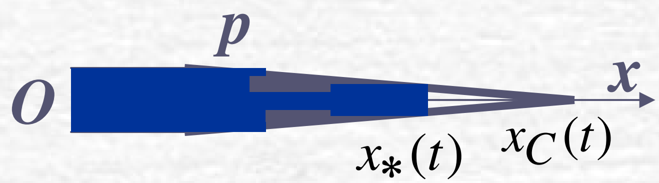
Savitski & Detournay 2002: **self-similar axisymmetric problem for Newtonian liquid;**

Michell, Kuske & Pierce 2007: **asymptotics and regimes**

Hu & Garagash 2010: **plane problem; accounting for leak-off**

Mathematical Formulation

Equations for Liquid



Fracture inlet $x = 0$

Liquid front $x_*(t)$ **Crack tip** $x_C(t)$

Lag $x_C - x_*$

Continuity equation (mass conservation)

$$div\boldsymbol{q} + \partial w / \partial t - q_e = 0 \qquad \textbf{(1)}$$

Poiseuille equation (viscous flow in narrow channel)

$$\boldsymbol{q} = -D(w, p)\text{grad}p \qquad \textbf{(2)}$$

Reynolds equation (using (2) in (1))

$$div[D(w, p)\text{grad}p] - \partial w / \partial t + q_e = 0$$

Initial condition (zero opening) $w(x,0) = 0$

Boundary condition (at liquid contour)

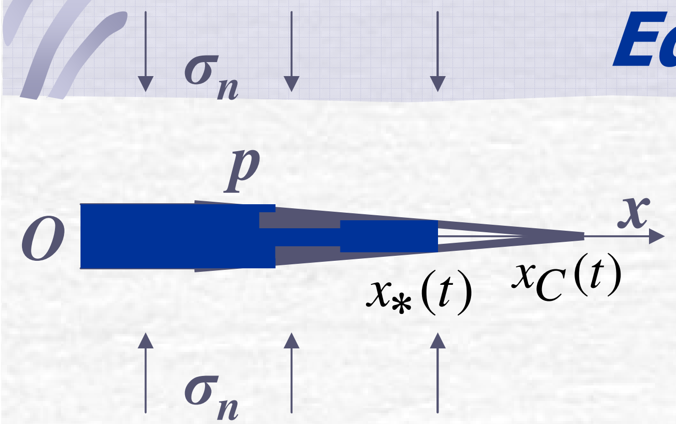
$$q_n(\boldsymbol{x}) = q_0(\boldsymbol{x}) \quad \boldsymbol{x} \in L_q \qquad p(\boldsymbol{x}) = p_0(\boldsymbol{x}) \quad \boldsymbol{x} \in L_p$$

Global mass balance $dV_e / dt = \int_{S_t} \left(\partial w / \partial t + \text{div}\boldsymbol{q} \right) dS$

The opening *w* being unknown, we need an equation for embedding solid (rock)

Mathematical Formulation

Equations for Solid



Fracture inlet $x = 0$
Liquid front $x_*(t)$ **Crack tip** $x_C(t)$
Lag $x_C - x_*$

Solid mechanics equation

(commonly BIE of linear elasticity)

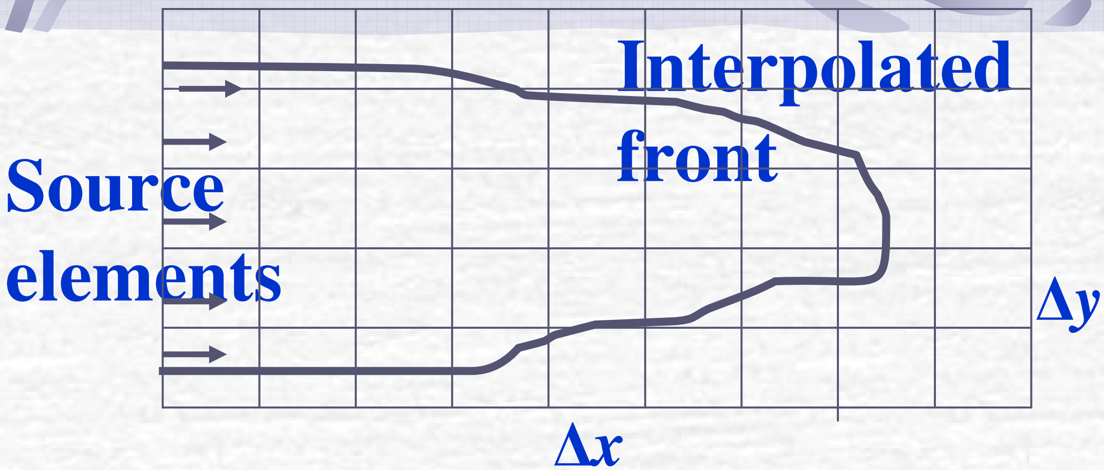
$$A(w, p) = 0$$

Boundary condition (at crack contour) $w(\boldsymbol{x_c}) = 0$

Fracture mechanics strength equation $K_I = K_{Ic}$
(commonly in terms of SIFs)

Strength limitation permits crack propagation; in general, it also defines the *lag* between the liquid front and the crack tip

Simulators of Hydraulic Fractures



Planar fracture geometry based on rectangular boundary elements

Simulators

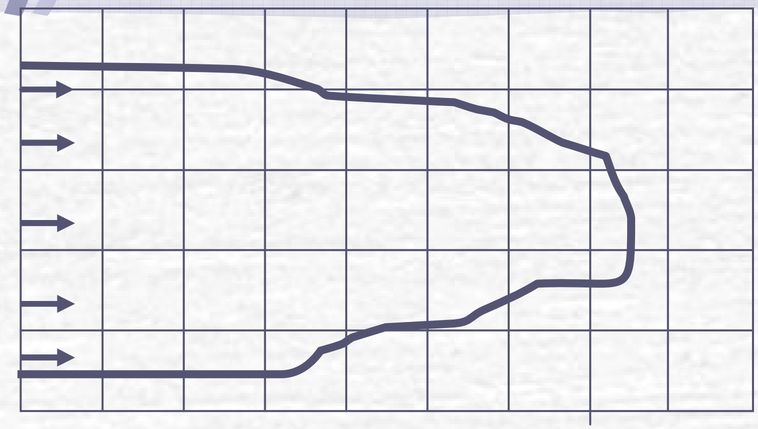
USA: Schlumberger (Siebrits et al)
USA: (Cleary et al)
Japan: (Jamamoto et al.)

}? Black boxes

Inexplicitly, numerics built in Schlumberger code is sketched in:
Adachi, Siebrits et al, *Int. J. Rock Mech Min. Sci.*, 2007, 44, 739-757

The authors emphasize the need
"to dramatically speed up … simulators"

Means to Meet Challenge



We need clear understanding of computational difficulties, which strongly influence the accuracy and stability of numerical results and robustness of procedures

An appropriate means may be:

Using the methods developed in well-established
THEORY OF PROPAGATING INTERFACES

J. A. Sethian, *Level Set Methods and Fast Marching Methods*,
Cambridge, Cambridge Univ. Press, 2nd ed., 1999

The basic concept of these methods is

SPEED FUNCTION

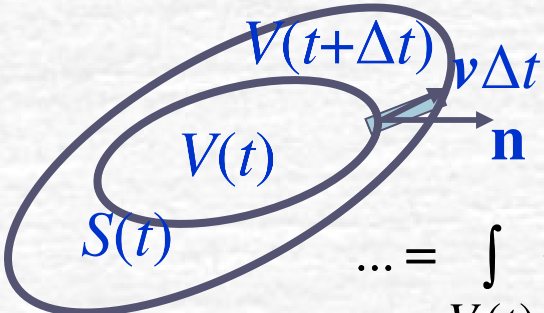
To the date, it has not been employed for hydraulic fracture simulation

Further discussion explains the reasons "WHY NOT?"

Speed Function and Speed Equation
for Hydraulic Fracture

REVISITING FUNDAMENTALS

Eqn for time derivative of an integral over moving volume



$\rho(x,t)$ arbitrary function

$$\frac{d}{dt}\int_{V(t)} \rho(\boldsymbol{x},t)\,dV = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int_{V+\Delta V}\rho(\boldsymbol{x},t+\Delta t)\,dV - \int_{V}\rho(\boldsymbol{x},t)\,dV\right\} =$$

$$\ldots = \int_{V(t)}\frac{\partial\rho}{\partial t}dV + \int_{S(t)}\rho v_n\,dS$$

v_n is the particle velocity at $S(t)$

Then the mass concervation law reads

$$\frac{dM_e}{dt} = \int_{V(t)}\frac{\partial\rho}{\partial t}dV + \int_{S(t)}\rho v_n\,dS \qquad \textbf{MC}$$

where now $\rho(x,t)$ is the mass density, *Me* is the external mass income

For ***incompressible homogeneous*** liquid $\rho(x,t)$ = const, $Me = \rho Ve$,

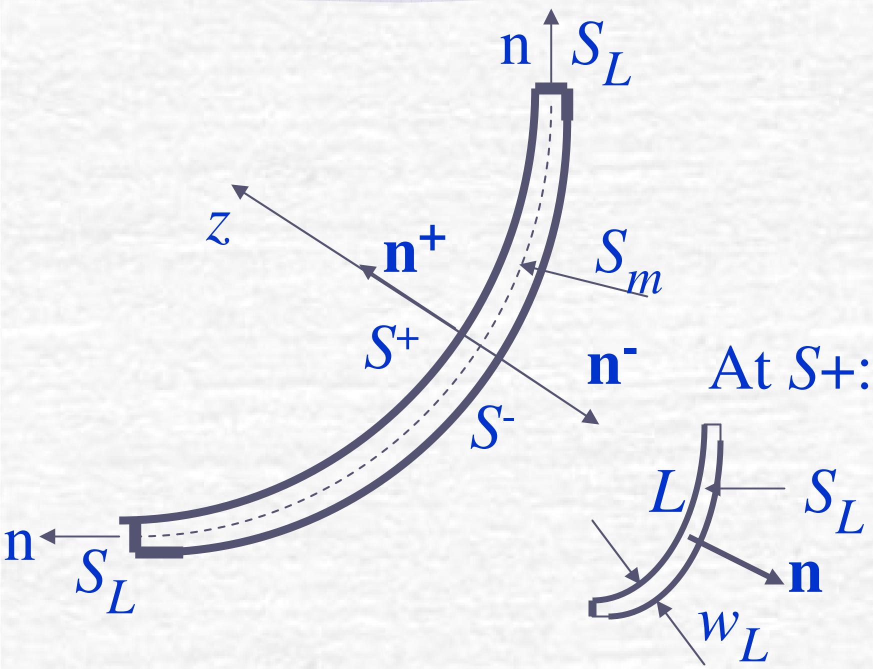
Volume conservation law (with obvious physical meaning):

$$\frac{dV_e}{dt} = \int_{S(t)} v_n\,dS \qquad \textbf{VC}$$

Specify the VC for flow in a narrow channel

Speed Function and Speed Equation

Volume conservation for flow in narrow channel



$$\frac{dV_e}{dt} = \int_{S(t)} v_n dS$$ **VC**

$S(t) = S^+ + S^- + S_L$

At $S+$: $v_n = \partial u_z^+ / \partial t$ At S^- $v_n = -\partial u_z^- / \partial t$

At S_L: $dS_L = w_L dL$

Hence for flow in narrow channel

$$\int_{S(t)} v_n dS = \int_{S_m(t)} \frac{\partial w}{\partial t} dS + \int_{L(t)} w v_n dS$$

where $w = u_z^+ - u_z^-$ is the channel width (opening)

VC equation becomes

$$\frac{dV_e}{dt} = \int_{S_m(t)} \frac{\partial w}{\partial t} dS + \int_{L(t)} q_n dL$$

Herein $q_n = w v_n$ ***is the total flux through the opening w***

Physical meaning is obvious. For a 'rigid-wall' channel, $\partial w / \partial t = 0$

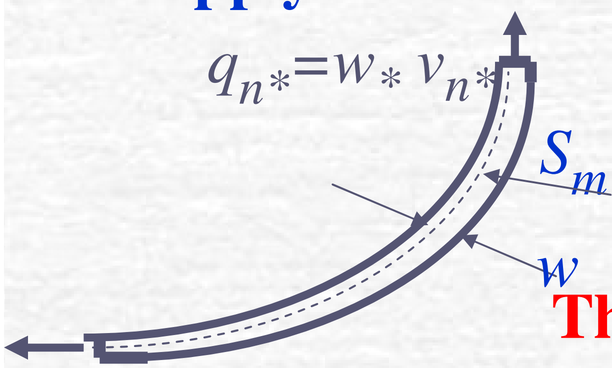
Speed Function and Speed Equation

VC for entire liquid in narrow channel

Thus for any volume of incompressible homogeneous liquid:

$$\frac{dV_e}{dt} = \int_{S_m(t)} \frac{\partial w}{\partial t} dS + \int_{L(t)} q_n dL \qquad \textbf{VC}$$

Apply it to the *entire* volume V_t occupied by liquid at time t



$$\frac{dV_e}{dt} = \int_{S_t} \frac{\partial w}{\partial t} dS + \int_{L_*(t)} q_{n*}(x_*) dL$$

$q_{n*} = w_* v_{n*}$ **flux through the liquid front**

The front Speed Equation is $v_{n*} = \frac{dx_{n*}}{dt} = \frac{q_{n*}}{w_*}$

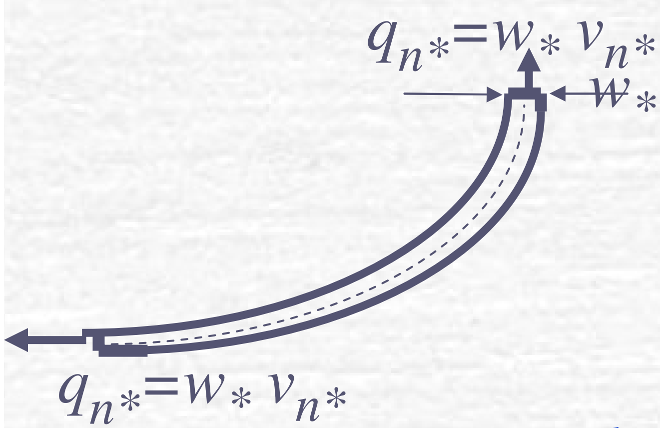
$F = \frac{q_{n*}}{w_*}$ **is the so-called Speed Function (SF)**

***Comment*. For 1-D case, VC yields** $\int_0^{x_*} w(x,t)dx = V_e(t)$ **Then for 'rigid' channel, integration gives** $x_*(t) = f(V_e(t))$ **This implies that** $x_* = \infty$ **at finite time t_* if width $w(x)$ decreases fast enough.**

Solution does not exist for $t > t_*$.

This simple example indicates possible difficulties for a narrow channel

Particular Forms of SE and SF:
flow in narrow channel



$$v_{n*} = \frac{q_{n*}}{w_*}$$ **Speed *Equation* (SE)**

$$F = \frac{q_{n*}}{w_*}$$ **Speed *Function* (SF)**

NOTE THAT

these forms of SE and SF are

specific for hydraulic fracture:

they appear ONLY because the channel is narrow

what gives rise to the concept

of the total flux q through the width of the channel

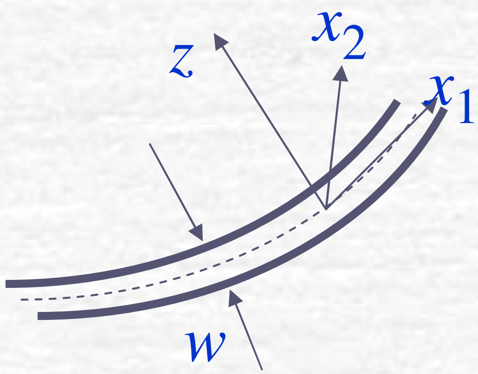
Respectively, only the total flux q enters Poiseuille equation

for viscous flow in a narrow channel

Comment. **The particle velocity $v = q/w$ behind the front does not enter HF equations. Still its clear physical meaning makes it of value for an appropriate choice of unknown functions**

Specification of Speed Function
for hydraulic fracture

Poiseuille eqn for flow of viscous liquid in narrow channel



$$\boldsymbol{q} = -D(w, p)\mathrm{grad}p$$

Vectors $q = (q_1, q_2)$ $grad = (\partial / dx_1, \partial / dx_2)$

are defined in the channel tangent plane

This yields the speed equation for hydraulic fracture front

$$v_{n*} = -\frac{1}{w_*} D(w, p) \frac{\partial p}{dn_*}$$

SPEED *EQUATION*
for hydraulic fracture

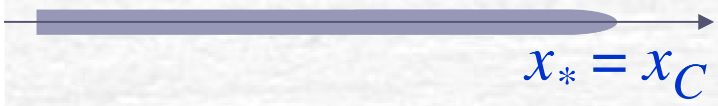
with

$$F = \frac{q_{n*}}{w_*} = -\frac{1}{w_*} D(w, p) \frac{\partial p}{dn_*}$$

SPEED *FUNCTION*
for hydraulic fracture

Specification of Speed Equation

for zero-lag case



Authors of computer simulators and commonly authors of papers on hydraulic fracture assume that there is *NO LAG* between the fluid front and the crack tip: $\boldsymbol{x}_* = \boldsymbol{x}_C$

Thus both the opening and the flux are zero at the crack contour coinciding with the liquid front

- $w_* = 0,$
- $q_{n*} = 0$

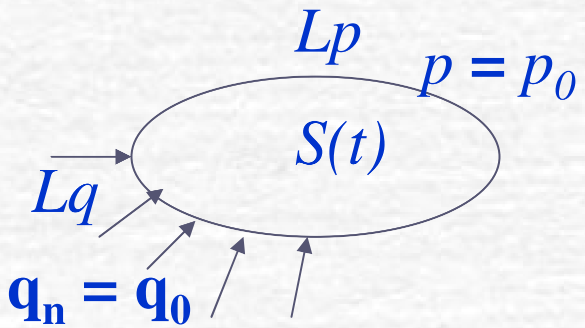
This results in the uncertainty $v_{n*} = \dfrac{q_*}{w_*} = \dfrac{0}{0}$ what complicates using of SF

Perhaps this explains why the speed equation has not been used for simulation of hydraulic fracture to the date

Still the speed equation is to be met in limit:

$$v_{n*}(\boldsymbol{x}_*) = \lim_{x \to x_*} \frac{q_n(\boldsymbol{x})}{w(\boldsymbol{x})} = \lim_{x \to x_*} \left[-\frac{1}{w(\boldsymbol{x})} D(w,p) \frac{\partial p}{\partial n_*} \right]$$

Particular Feature of Problem
for hydraulic fracture



Continuity equation (local form)

$$div\boldsymbol{q} + \partial w / \partial t - q_e = 0 \qquad \textbf{(1)}$$

Poiseuille equation

$$\boldsymbol{q} = -D(w, p)\mathrm{grad}p \qquad \textbf{(2)}$$

Reynolds equation (using (2) in (1))

$$div[D(w, p)\mathrm{grad}p] - \partial w / \partial t + q_e = 0$$

Initial condition (zero opening) $w(x,0) = 0$

BC from physical considerations (at the liquid contour)

$$q_n(\boldsymbol{x}) = q_0(\boldsymbol{x}) \quad \boldsymbol{x} \in L_q \qquad p(\boldsymbol{x}) = p_0(\boldsymbol{x}) \quad \boldsymbol{x} \in L_p \qquad \textbf{BC}$$

But !We have additional SPEED EQUATION (at the liquid contour)

$$v_{n*} = \frac{q_*}{w_*} = -\frac{1}{w_*} D(w, p) \frac{\partial p}{dn_*} \qquad x \in L_q + L_p \quad \textbf{BC=SE}$$

Thus for the *elliptic* (in spatial coordinates) *operator*
we have *two* rather than one *boundary conditions*
involving *a function and normal derivative*.
This indicates that there might be difficulties.
Specifically, a problem might be *ill-posed*

Hadamard Definition and Tychonoff Regularization

Jacuques Hadamard **(1902),** ***Sur les problemes aux derivees partielles et leur signification physique*****, Princeton University Bulletin 49-52**

By Hadamard, a problem is *well-posed* when

- **A solution exists**
- **The solution is unique**
- **The solution depends continuously on the data,**

in some reasonable metric

Otherwise, a problem is *ill-posed*

A.N. Tychonoff **(1943) clearly recognized significance of ill-posed problems for applications. He was the first to suggest a means to solve them numerically by using *regularization:***

A.N. Tychonoff (1963) ***Solution of incorrectly formulated problems and the regularization method*****, Soviet Mathematics 4, 1035-1038. [Transl. from Russian: А. Н. Тихонов, ДАН СССР, 1963, 151, 501-504]**

Need in Clear Example

to display difficulties and suggest regularization

It looks reasonable to illustrate the specific features of the hydraulic fracture simulation by a clear example:

“The art of doing mathematics consists in finding that special case which contains all the germs of generality.”

D. Hilbert

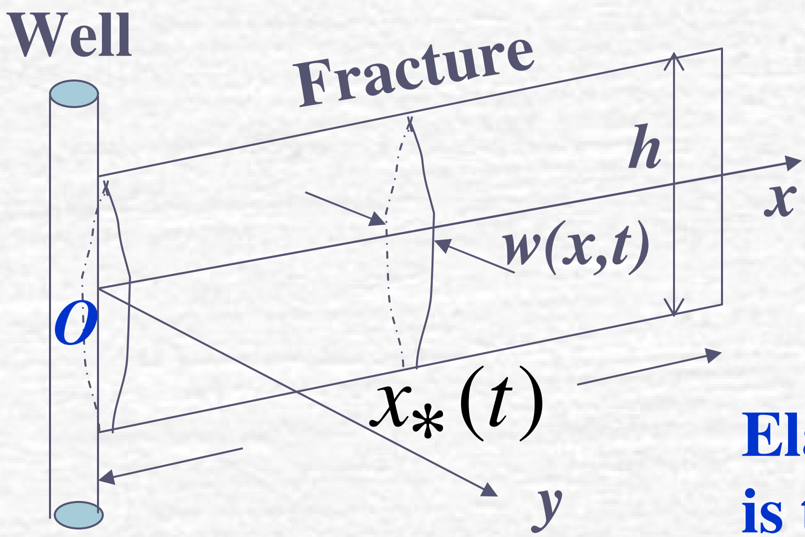
Quoted in: N. Rose *Mathematical Maxims and Minims*, Raleigh N. C., 1988

Consider the Nordgren problem as *“that special case”*

- **to evidently see that the problem is ill-posed and**
- **to find a proper method of regularization to have accurate and stable numerical results**

Nordgren Problem

formulation



Continuity equation (no leak-off)

$$\partial q / \partial x + \partial w / \partial t = 0$$

Poiseuille equation (Newtonian liquid)

$$q = -k_l w^3 \partial p / \partial x$$

Reynolds equation (Newtonian liquid)

$$k_l \frac{\partial}{\partial x}\left(w^3 \frac{\partial p}{\partial x} \right) - \frac{\partial w}{\partial t} = 0$$

Elasticity equation to find *w* is taken in the simplest form

$$p = k_r w$$

Then after using dimensionless variables, the problem becomes

$$\frac{\partial^2 w^4}{\partial x^2} - \frac{\partial w}{\partial t} = 0$$ Nordgren PDE

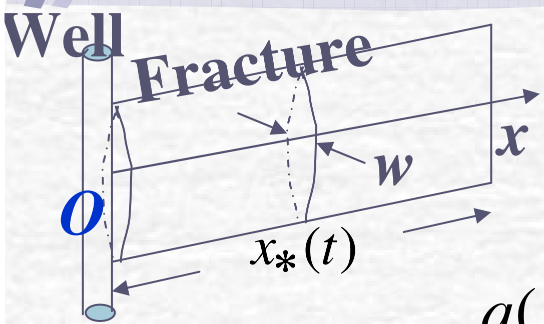
$$-\left.\frac{\partial w^4}{\partial x}\right|_{x=0} = \boldsymbol{q}_0$$ BC at inlet $x = 0$

$$w(\boldsymbol{x}_*, t) = 0$$ BC at liquid front $x = x_*$

The solution should be such that:

$$w(\boldsymbol{x}, t) > 0 \quad 0 \le x < \boldsymbol{x}_*, \qquad w(\boldsymbol{x}, t) = 0 \quad x > \boldsymbol{x}_*$$

Speed Equation for Nordgren Problem
reformulation in terms of w^3



Nordgren solved the problem in *w* by Crank-Nicolson method to the accuracy of 1%

Global balance or speed equation were not used

The speed equation (*no-lag case*) is

$$v_* = \lim_{x \to x_*} \frac{q(x)}{w(x)}$$ **where now** $q = -\frac{\partial w^4}{\partial x}$ **Then** $\frac{q}{w} = -\frac{4}{3}\frac{\partial w^3}{\partial x}$

and the speed equation becomes $$v_* = -\frac{4}{3}\frac{\partial w^3}{\partial x}\bigg|_{x=x_*}$$ **SE**

The SE implies that the variable w^3 is preferable: its spatial derivative is finite. In terms of w^3, the problem is reformulated as

$$\frac{\partial^2 w^3}{\partial x^2} + \frac{1}{3w^3}\left(\frac{\partial w^3}{\partial x}\right)^2 - \frac{1}{4w^3}\frac{\partial w^3}{\partial t} = 0$$ **PDE**

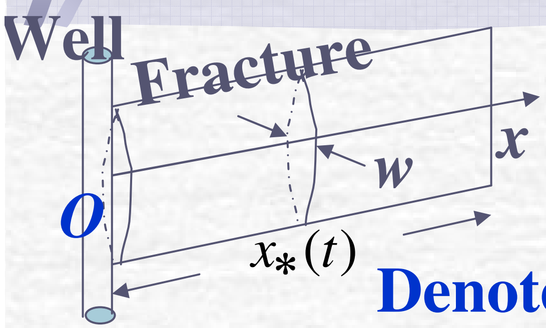
$$\frac{dw^3}{\partial x}\bigg|_{x=0} = -0.75\frac{q_0}{\sqrt[3]{w^3(0)}}$$ **BC at inlet $x = 0$**

$$w^3(x_*) = 0$$ **BC at liquid front $x = x_*$**

+ SPEED EQUATION at the liquid front

Self-Similar Formulation

evidence that the problem is ill-posed



The problem is self-similar. Introduce automodel variables

$$x = \xi t^{4/5}, \quad w(x) = t^{1/5}\psi(xt^{-4/5}) \quad x_* = \xi_* t^{4/5}$$

Denote $y(\xi) = \psi^3(\xi)$ **The problem is reduced to ODE**

$$\frac{d^2 y}{d\xi^2} + a(y, dy/d\xi, \xi)\frac{dy}{\partial\xi} - \frac{3}{20} = 0$$ **ODE**

where $a(y, dy/d\xi, \xi) = (dy/d\xi + 0.6\xi)/(3y)$ **is finite at liquid front** $\xi = \xi_*$

$$\left.\frac{dy}{\partial\xi}\right|_{\xi=0} = -0.75\frac{q_0}{\sqrt[3]{y(0)}}$$ **BC at inlet $x = 0$**

$$y(\xi_*) = 0$$ **BC at liquid front $x = x_*$**

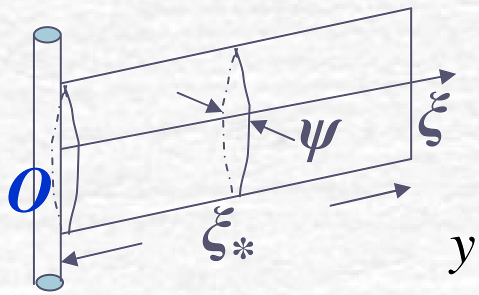
+ *SPEED EQUATION at the liquid front*

$$\left.\frac{dy}{\partial\xi}\right|_{\xi=\xi_*} = -0.6\xi_*$$ **SE**

Evidently, there are *two* BC at the liquid front

Invariant Constants of N-Problem

option to fix ξ_*



By direct substitution, it is easy to check that:

If $y_1(\xi_1)$ is a solution for $q = q_{01}$ with $\xi_* = \xi_{*1}$ then

$y_2(\xi_2) = y_1(\xi_2\sqrt{k})/k$ is a solution for $q_{02} = k^{-5/6}q_{01}$

with $\xi_{*2} = \xi_{*1}/\sqrt{k}$ $\quad$ k is an arbitrary positive number

This implies that there are two constants not depending on q_0:

$$C_* = (q_0)^{0.6}/\xi_* \qquad C_0 = y(0)/\xi_*^2$$

Conclusions:

❖ **Since**

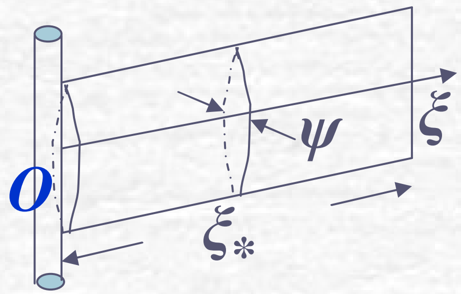
$$\xi_* = (q_0)^{0.6}/C_*$$

it is a matter of convenience to prescribe q_0 or ξ_*

❖ **A particular value of ξ_* may be also taken as convenient**

Ill-Posed Boundary Value Problem

versus *well-posed initial value problem*



Thus we have the problem for ODE

$$\frac{d^2 y}{d\xi^2} + a(y, dy/d\xi, \xi)\frac{dy}{\partial\xi} - \frac{3}{20} = 0 \qquad \textbf{ODE (1)}$$

with conditions: $\left.\frac{dy}{\partial\xi}\right|_{\xi=0} = -0.75\frac{q_0}{\sqrt[3]{y(0)}}$ **BC at inlet** $\xi = 0$ **(2)**

$y(\xi_*) = 0$ **BC at liquid front** ξ_* **(3)**

$\left.\frac{dy}{\partial\xi}\right|_{\xi=\xi_*} = -0.6\xi_*$ **SE = BC at liquid front** ξ_* **(4)**

where ξ_* is prescribed. For certainty, $\xi_* = 1$.

Hence, for ODE of second order, at the point $\xi_* = 1$, we have prescribed both the function and its derivative. Its solution defines q_0. Therefore, a small error in prescribing q_0 excludes the solution of the BV problem (1) - (3). By Hadamard definition

BV problem (1) - (3) is ill-posed. ***IV problem (1), (3), (4) is well-posed***

Bench-Mark Solution
of *well-posed initial value problem*

Initial value (Cauchy) problem for ODE

$$\frac{d^2 y}{d\xi^2} + a(y, dy/d\xi, \xi)\frac{dy}{\partial\xi} - \frac{3}{20} = 0$$ **ODE (1)**

with *initial* conditions:

$$y(\xi_*) = 0$$ **BC at front (3)**

$$\left.\frac{dy}{\partial\xi}\right|_{\xi=\xi_*} = -0.6\xi_*$$ **BC at front (4)**

The problem is solved by using R-K scheme of forth order

The bench-mark solution is obtained with 7 correct digits

$$C_* = 0.7570913 \qquad C_0 = 0.5820636$$

Values of $\psi_1 = \sqrt[3]{y_1}$ **and** $d\psi_1^3 / d\xi_1$ **are tabulated for** $\xi_* = \xi_{*1} = 1$

For the value $q_0 = 2/\pi$ **used by Nordgren, the benchmarks are**

$$\xi_* = 1.007348 \qquad \psi(0) = 0.839028$$

against the values given by Nordgren to the accuracy of about 1%

$$\xi_* = 1.01 \qquad \psi(0) = 0.83$$

The bench-mark solution serves us to evaluate the accuracy of calculations without and with regularization

Solution of Self-Similar BV Problem
without regularization

Ill-posed BV problem:

$$\frac{d^2 y}{d\xi^2} + a(y, dy/d\xi, \xi)\frac{dy}{\partial \xi} - \frac{3}{20} = 0$$ **ODE (1)**

with *boundary* conditions:

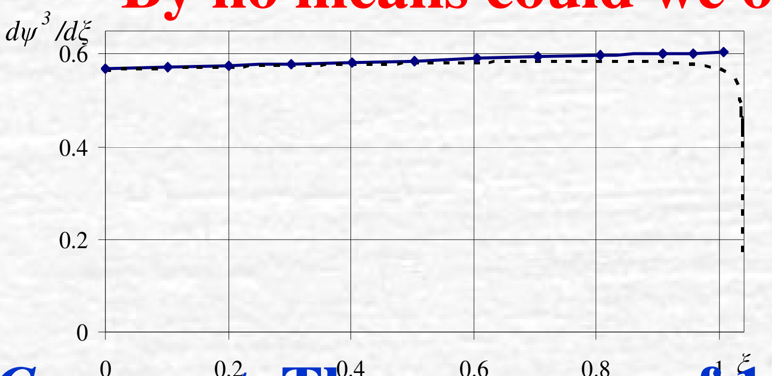
$$\left.\frac{dy}{\partial \xi}\right|_{\xi=0} = -0.75\frac{q_0}{\sqrt[3]{y(0)}}$$ **BC at inlet (2)**

$$y(\xi_*) = 0$$ **BC at front (3)**

Up to 100 000 nodal points and up to 1500 iterations were used in attempts to reach the accuracy of three correct digits, at least

THE ATTEMPTS HAVE FAILED

By no means could we obtain more than two correct digits



The results always deteriorate near the liquid front

Comment.* The accuracy of 1% is obtained even when using a rough mesh. Thus a rough mesh may serve for regularization when high accuracy is of no need. *Still we need an appropriate regularization

Solution of Self-Similar BV Problem

with ε-regularization

From the BC and SE at the front it follows that near the front: $y \approx 0.6\xi_*(\xi_* - \xi)$

Hence instead of prescribing a BC at the front, we may impose it at a point at a small relative distance ε behind the front as

$$y(\xi_\varepsilon) = 0.6\xi_*^2\varepsilon$$

Now the problem is *well-posed*. It is solved by finite differences with iterations in non-linear terms and ξ_*.

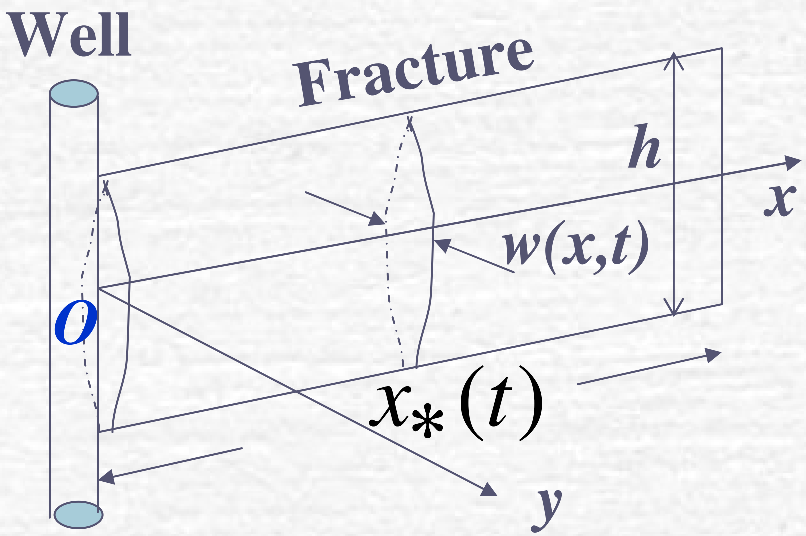
For $\varepsilon = 10^{-3}, 10^{-4}$, the results for the step $\Delta\varsigma = \Delta\xi / \xi_* = 10^{-3} - 10^{-6}$ coincide with the bench-mark solution

The essence of the suggested regularization consists in using the speed equation together with a prescribed BC to formulate the BC at a small relative distance ε behind the front rather than on the front itself

We call such an approach ε - regularization

***Comment*. For a coarse mesh the accuracy actually does not depend on the regularization parameter ε**

Solution of Starting Problem *without* regularization



$$\frac{\partial^2 w^4}{\partial x^2} - \frac{\partial w}{\partial t} = 0$$ **Nordgren PDE**

$$-\left.\frac{\partial w^4}{\partial x}\right|_{x=0} = q_0$$ **BC at inlet**

$$w(x_*, t) = 0$$ **BC at front**

We solved the starting N-problem by using Crank-Nicolson scheme *without regularization*

By no means could we have more than two correct digits

Similar to self-similar solution, fine meshes gave no improvement of the accuracy as compared with a rough mesh

When using as unknowns w^4 and w^3, the conclusions are same

Solution of Starting Problem

with ε – regularization: change of spatial coordinate

$$\frac{\partial^2 w^3}{\partial x^2} + \frac{1}{3w^3}\left(\frac{\partial w^3}{\partial x}\right)^2 - \frac{1}{4w^3}\frac{\partial w^3}{\partial t} = 0$$ **PDE**

Introduce the coordinate moving with the front

$$\varsigma = x / x_*(t)$$

Recalculate partial derivative $\partial\varphi/\partial t\big|_{x=const}$

to partial derivative $\partial\Phi/\partial t\big|_{\varsigma=const}$ **with** $\Phi(\varsigma,t) = \varphi(\varsigma x_*(t),t)$

$$\left.\frac{\partial\varphi}{\partial t}\right|_{x=const} = \left.\frac{\partial\Phi}{\partial t}\right|_{\varsigma=const} - \varsigma\frac{v_*(t)}{x_*(t)}\frac{\partial\Phi}{\partial\varsigma}$$

The PDF becomes $$\frac{\partial^2 Y}{\partial\varsigma^2} + A(Y,\partial Y/\partial\varsigma, x_* v_* \varsigma)\frac{\partial Y}{\partial\varsigma} - B(Y,x_*)\frac{\partial Y}{\partial t} = 0$$

where

$$Y(\varsigma,t) = w^3(\varsigma x_*(t),t) \quad A(Y,\partial Y/\partial\varsigma, x_* v_* \varsigma) = \frac{\partial Y/\partial\varsigma + 0.75 x_* v_* \varsigma}{3Y} \quad B(Y,x_*) = \frac{x_*^2}{4Y}$$

BC at inlet $$\left.-\frac{4}{3}\frac{\sqrt[3]{Y}}{x_*}\frac{\partial Y}{\partial\varsigma}\right|_{\varsigma=0} = q_0$$

+ SE at front $$\left.\frac{\partial Y}{\partial\varsigma}\right|_{\varsigma=1} = -0.75 x_* v_*$$

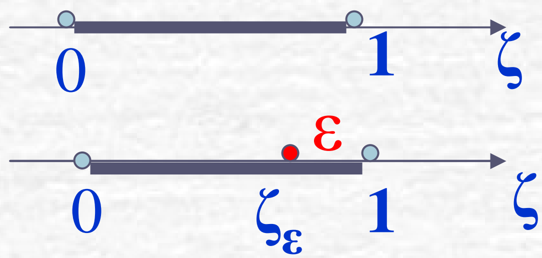
BC at front $$Y(\varsigma,t)\big|_{\varsigma=1} = 0$$

From conditions on the front it follows that near the front:

$$Y(\varsigma,t) \approx 0.75 x_*(t) v_*(t)(1-\varsigma)$$

Solution of Starting Problem

ε – regularization



We have obtained that near the front:

$$Y(\varsigma, t) \approx 0.75 x_*(t) v_*(t)(1-\varsigma)$$

Hence, now we may impose the BC at the relative distance ε behind the front

$$Y(\varsigma_\varepsilon, t) = 0.75 x_*(t) v_*(t)\varepsilon$$ **BC *near* front**

We may expect that the problem is well-posed and provides the needed regularization. Extensive numerical tests confirm the expectation

The problem is solved by using Crank-Nicolson scheme and iterations for non-linear multipliers and $v_*(t)$

For $10^{-2} > \varepsilon > 10^{-4}$ $\Delta\varsigma \le 0.01$ **the results are accurate and stable in a wide range of the values of the time step and for very large number (up to 100 000) of steps. Error is less than 0.03%. There are no signs of instability in specially designed experiments**

The time expense on a conventional laptop does not exceed 15 s

This implies that ε - regularization is efficient

Conclusions

- The *speed function* for fluid flow in a thin channel is given by the ratio of the total flux to width at the front. Its using facilitates employing level set methods and fast marching methods.
- The *speed equation* is a general condition at the liquid front *additional* to commonly formulated BC for hydraulic fracture.
- Using the SE extends options for numerical simulation of HF. It also indicates that the problem may be *ill-posed*.
- Suggested **ε** - *regularization* consists in employing the SE with a prescribed BC on the front to get a new BC at a small distance behind the front. It appears to be efficient.
- Studying the Nordgren problem evidently discloses the features of hydraulic fracture simulation. It gives the key to overcome the difficulty. Its solution provides the *bench-marks* useful for evaluating the accuracy.

Thank you!