XXXX Summer School

ADVANCED PROBLEMS IN MECHANICS

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# Evaluation of temporal derivative for propagating front of hydraulic fracture

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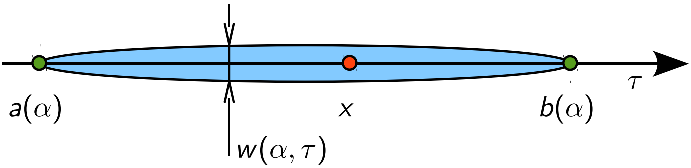
## Hydraulic Fracturing

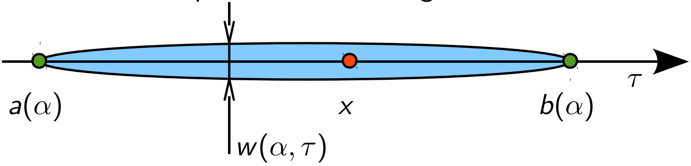
The work concerns with the problem of hydraulic fracture propagating in time.




$\alpha$ is the time (parameter).

## Problem formulation

2D problem for a straight crack.




The classical elasticity equation connecting the net-pressure $p$ and the opening $w$ is:

$$p(\alpha, x) = -\frac{E}{4\pi(1-\nu^2)} \int_{a(\alpha)}^{b(\alpha)} \frac{\partial w(\alpha, \tau)}{\partial \tau} \frac{d\tau}{\tau - x}, \qquad a \le x \le b,$$

where $E$ is the elasticity modulus, $\nu$ is the Poisson's ratio of the rock mass.

## Problem formulation

A corresponding hypersingular form

$$p(\alpha, x) = -\frac{E}{4\pi(1-\nu^2)} \int_{a(\alpha)}^{b(\alpha)} \frac{w(\alpha,\tau)d\tau}{(\tau - x)^2}.$$

The rate of the pressure change $\frac{\partial}{\partial\alpha}p(\alpha, x)$ is a characteristic strongly dependent on the fluid injection regime.

**The integral on the r. h. s. is hypersingular.**

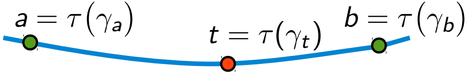
**The question is: how to differentiate it**

**with respect to the parameter $\alpha$ (time)?**

We need to **extend the theory** to **obtain the differentiation rule**.

# Complex variable hypersingular integrals

The CV hypersingular integral




$$I_k(t) = \int_a^b \frac{g(\tau)}{(\tau - t)^k} d\tau, \text{ where } k \geqslant 1,$$

defined in accordance with the general theory
refers to moment, it then refers to problems in which the boundary
of a surface is fixed.

See e.g. Linkov A. M., *Boundary Integral Equations in Elasticity Theory*, Dordrecht, Kluwer Academic Publishers, 2002.

## Special cases of CVHI

- For $k = 1$ we have Cauchy integral

$$I_1(t) = \int_a^b \frac{g(\tau)}{\tau - t} d\tau,$$

- For $k = 2$ Hadamard integral

$$I_2(t) = \int_a^b \frac{g(\tau)}{(\tau - t)^2} d\tau.$$

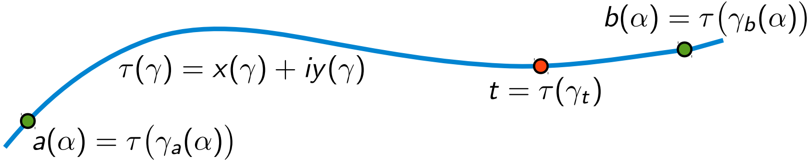
## Problems to be discussed

- concept of the CVHI with the density and limits of integration depending on a parameter,
- extension to a density with derivative(s) having power-type singularity at arc tips.

## The definition of the CVHI of order $k$ with a parameter

$$I_k(\alpha, t) = \int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha, \tau)}{(\tau - t)^k} d\tau.$$

- $ab$ is an open, smooth curve (arc) in the complex plane,
- $\gamma_a(\alpha)$, $\gamma_b(\alpha)$ have Holder continuous derivatives,
- the density $g(\alpha, \tau)$ has Holder continuous $\frac{\partial^k g(\alpha,\tau)}{\partial \tau^k}$ and also Holder continuous $\frac{\partial g(\alpha,\tau)}{\partial \alpha}$.



## Basic properties

Two useful formulae:

- Extended Newton-Leibnitz formula:

$$\int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha,\tau)}{(\tau-t)^k}d\tau = J_g(\alpha,b) - J_g(\alpha,a) + \frac{i\pi}{k!}g_t^{(k-1)}(\alpha,t),$$

where $J_g(\alpha,\tau)$ is an antiderivative of the integrand $\frac{g(\alpha,\tau)}{(\tau-t)^k}$.

- The regularization formula for $k \geq 2$:

$$\frac{d}{dt}\int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha,\tau)}{(\tau-t)^{k-1}}d\tau = (k-1)\int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha,\tau)}{(\tau-t)^k}d\tau.$$

# Differentiation of a CVHI with respect to a parameter

**Theorem**

*The derivative of a hypersingular integral*

$$I_k(\alpha, t) = \int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha, \tau)}{(\tau - t)^k} d\tau$$

*with respect to the parameter* $\alpha$ *may be evaluated as*

$$\frac{\partial I_k(\alpha, t)}{\partial \alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial g(\alpha, \tau)}{\partial \alpha} \frac{d\tau}{(\tau - t)^k} + \frac{g(\alpha, b)}{(b - t)^k} \frac{db}{d\alpha} - \frac{g(\alpha, a)}{(a - t)^k} \frac{da}{d\alpha}.$$

**We can see that the theorem is the same as the well known formula for a proper integral.**

## Density with derivatives having power-type singularity

Density of the form $g(\alpha,\tau)=(c-\tau)^{\gamma}g_{\gamma}(\alpha,\tau)$, $c=a$ or $c=b$.

If $j-1<Re\gamma<j$, then the derivatives $\frac{\partial^j g(\alpha,\tau)}{\partial\tau^j}$ and $\frac{\partial^j g(c,\tau)}{\partial\tau^{j-1}\partial c}$ are singular at the point $\tau=c$, tending to infinity as $\frac{1}{(c-\tau)^{j-Re\gamma}}$.

Thus the integral

$$\int_{a(\alpha)}^{b(\alpha)}\frac{g(\alpha,\tau)}{(\tau-t)^k}d\tau$$

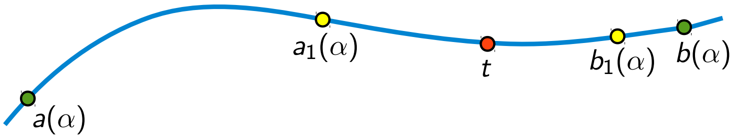
of order $k=j+1$ with such density is not defined.

**Density derivatives aren't Holder continuous!**

## Differentiation of a integral with the density which derivatives have power-type singularity

For such a density we may represent the integral as the sum:

$$\int_{a(\alpha)}^{a_1(\alpha)} \frac{g(\alpha,\tau)}{(\tau-t)^k} d\tau + \int_{a_1(\alpha)}^{b_1(\alpha)} \frac{g(\alpha,\tau)}{(\tau-t)^k} d\tau + \int_{b_1(\alpha)}^{b(\alpha)} \frac{g(\alpha,\tau)}{(\tau-t)^k} d\tau,$$




and the differentiation theorem holds for points within an arc $ab$.

**The second integral is hypersingular and is well defined.**

**The first and the third don't have any singularity!**

## Special case, when $g(\alpha, c) = 0$

When the density *(opening)* is zero at the edge points *(fracture front)* the differentiation formula means that it is possible to differentiate under the integral sign:

$$\frac{\partial}{\partial \alpha} \int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha,\tau)}{(\tau - t)^k} d\tau = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial g(\alpha,\tau)}{\partial \alpha} \frac{d\tau}{(\tau - t)^k}.$$

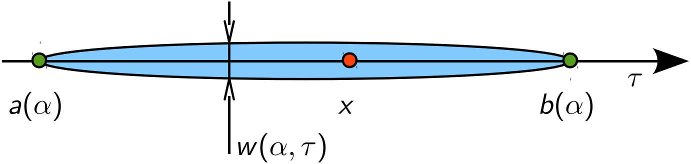
Using the regularization formula this equation may be written as

$$\frac{\partial}{\partial \alpha} \int_{a(\alpha)}^{b(\alpha)} \frac{g(\alpha,\tau)}{(\tau - t)^k} d\tau = \frac{1}{k-1} \frac{\partial}{\partial t} \int_{a(\alpha)}^{b(\alpha)} \frac{\partial g(\alpha,\tau)}{\partial \alpha} \frac{d\tau}{(\tau - t)^{k-1}}.$$

## Example – an early stage of the hydraulic fracturing:

- influence of viscosity is negligible,
- the net-pressure is constant along the fracture:
  $p(\alpha, x) = p(\alpha)$, $\partial p/\partial x = 0$,
- plain-strain conditions: the opening $w$ is given by the well-known formula

$$w(\alpha, \tau) = \frac{4(1-\nu^2)}{E} p(\alpha) \sqrt{(\tau - a(\alpha))(b(\alpha) - \tau)}.$$



The dependence between the net-pressure $p$ and the fracture opening $w$ in hypersingular form is:

$$p(\alpha) = -\frac{1}{\pi} \int_{a(\alpha)}^{b(\alpha)} \frac{p(\alpha)\sqrt{\left(\tau - a(\alpha)\right)\left(b(\alpha) - \tau\right)}d\tau}{(\tau - x)^2}.$$

For the derivative $\partial p/\partial\alpha = dp/d\alpha$, it yields

$$\frac{dp}{d\alpha} = -\frac{1}{\pi}\frac{\partial}{\partial\alpha} \int_{a(\alpha)}^{b(\alpha)} \frac{p(\alpha)\sqrt{\left(\tau - a(\alpha)\right)\left(b(\alpha) - \tau\right)}d\tau}{(\tau - x)^2}.$$

**We will evaluate the right hand side of this equation and compare the result with the left.**

From the differentiation rule, when $g(\alpha, c) = 0$, we have:

$$\frac{\partial}{\partial \alpha} \int_{a(\alpha)}^{b(\alpha)} \frac{p(\alpha)\sqrt{\left(\tau - a(\alpha)\right)\left(b(\alpha) - \tau\right)}}{(\tau - x)^2} d\tau =$$

$$= \frac{dp(\alpha)}{d\alpha} \int_{a(\alpha)}^{b(\alpha)} \frac{\sqrt{\left(\tau - a(\alpha)\right)\left(b(\alpha) - \tau\right)} d\tau}{(\tau - x)^2} +$$

$$+p(\alpha)\frac{1}{2}\frac{\partial}{\partial x} \int\limits_{a}^{b} \frac{(\tau - a)db/d\alpha - (b - \tau)da/d\alpha}{\sqrt{(\tau - a)(b - \tau)}(\tau - x)} d\tau.$$

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$$\frac{\partial}{\partial\alpha}\int_{a(\alpha)}^{b(\alpha)} \frac{p(\alpha)\sqrt{\left(\tau - a(\alpha)\right)\left(b(\alpha) - \tau\right)}}{(\tau - x)^2} d\tau =$$

$$= \frac{dp(\alpha)}{d\alpha} \underbrace{\int_{a(\alpha)}^{b(\alpha)} \frac{\sqrt{\left(\tau - a(\alpha)\right)\left(b(\alpha) - \tau\right)} d\tau}{(\tau - x)^2}}_{-\pi} +$$

$$+p(\alpha)\frac{1}{2}\frac{\partial}{\partial x}\int_{a}^{b} \frac{(\tau - a)db/d\alpha - (b - \tau)da/d\alpha}{\sqrt{(\tau - a)(b - \tau)}(\tau - x)} d\tau.$$

The general theory implies that $\overbrace{\int\limits_a^b \frac{d\tau}{\sqrt{(\tau-a)(b-\tau)}(\tau-x)}}^{(\star)} = 0,$

$$p(\alpha)\frac{1}{2}\frac{\partial}{\partial x}\int\limits_a^b \frac{(\tau-a)db/d\alpha-(b-\tau)da/d\alpha}{\sqrt{(\tau-a)(b-\tau)}(\tau-x)}d\tau$$

$$= p(\alpha)\frac{1}{2}\left(\frac{db}{d\alpha}+\frac{da}{d\alpha}\right)\frac{\partial}{\partial x}\int\limits_a^b \frac{\tau}{\sqrt{(\tau-a)(b-\tau)}(\tau-x)}d\tau.$$

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$$= p(\alpha)\frac{1}{2}\left(\frac{db}{d\alpha}+\frac{da}{d\alpha}\right)\frac{\partial}{\partial x}\int_a^b \frac{1}{\sqrt{(\tau-a)(b-\tau)}}d\tau$$

$$= 0.$$

That is:

$$\overbrace{\frac{\partial}{\partial \alpha} \int\limits_{a(\alpha)}^{b(\alpha)} \frac{p(\alpha)\sqrt{\left(\tau - a(\alpha)\right)\left(b(\alpha) - \tau\right)}d\tau}{(\tau - x)^2} = -\pi \frac{dp}{d\alpha}}^{(\star)},$$

what concludes the derivation and conforms the theorem:

$$\frac{dp}{d\alpha} = -\frac{1}{\pi}\frac{\partial}{\partial \alpha} \int\limits_{a(\alpha)}^{b(\alpha)} \frac{p(\alpha)\sqrt{\left(\tau - a(\alpha)\right)\left(b(\alpha) - \tau\right)}d\tau}{(\tau - x)^2}$$

$$\overset{(\star)}{=} -\frac{1}{\pi}\left( -\pi \frac{dp}{d\alpha}\right) = \frac{dp}{d\alpha}.$$

**Thank you for attention!**