Crack propagation in heterogeneous materials with several defects

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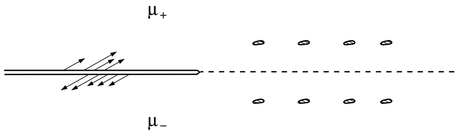
The problem and motivation



Crack advancing in a bimaterial plane with finite array of defects

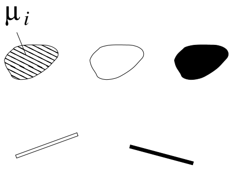
- What is the effect of the defects on the propagation of the crack? Amplification/shielding effects?
- Can we arrange the defects in such a way to stop the propagation of the crack?
- Vice versa, can we arrange the defects in such a way to make the crack propagate up to the end of the array?

New challenges:

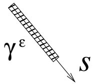
- Interfacial crack
- Weight function for an interfacial crack
- Singular perturbation procedure related to small defects
- Singular perturbation procedure related to small advance of the crack

Type of defects

- Small elastic inclusions
- Small voids
- Small rigid inclusions
- Microcracks
- Rigid line inclusions



Line defect with imperfect bonding:



- Line defect with soft bonding (stiffness κ): $[\![\sigma]\!](s) = 0, \quad \sigma(s) = \kappa [\![u]\!](s)$

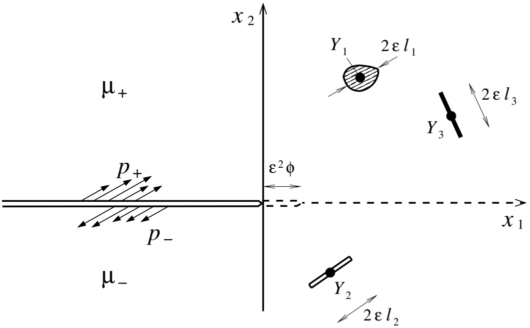
$$\begin{cases} \kappa = 0 & \Rightarrow \quad \sigma(s) = 0 \quad \text{microcrack} \\ \kappa = \infty & \Rightarrow \quad [\![u]\!](s) = 0 \quad \text{perfect bonding (no defect)} \end{cases}$$

- Stiff line defect (stiffness κ): $[\![u]\!](s) = 0, \quad [\![\sigma]\!](s) + \kappa \left. \frac{\partial^2 u}{\partial s^2} \right|_{\gamma^\epsilon} = 0$

$$\begin{cases} \kappa = 0 & \Rightarrow \quad [\![\sigma]\!](s) = 0 \quad \text{no defect} \\ \kappa = \infty & \Rightarrow \quad \left. \frac{\partial^2 u}{\partial s^2} \right|_{\gamma^\epsilon} = 0 \quad \text{rigid line inclusion} \end{cases}$$

Problem formulation

Bimaterial plane with a dominant crack along the interface and small defects:



Singular perturbations:

1. Elastic inclusion
2. Microcrack
3. Rigid line inclusion

Singular perturbation:

1. Crack advance

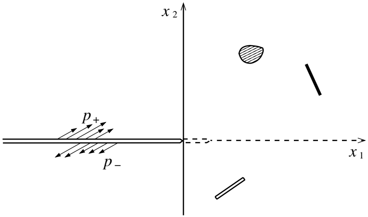
Small parameter ε:
diameter of defect $2\varepsilon l$, $\varepsilon \ll 1$

Assumptions:

1. Mode III deformation
2. Loading on the crack surfaces
3. Perfect interfaces (continuity of displacements and tractions)
4. The composite is dilute (neglect interactions between small defects)
5. Stable quasi-static propagation (neglect inertia terms)

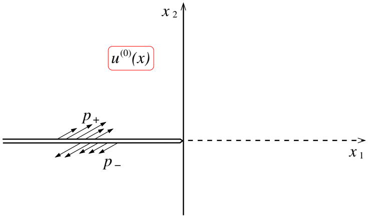
The asymptotic ansatz

$$u(\boldsymbol{x},\varepsilon) = u^{(0)}(\boldsymbol{x}) + \varepsilon \sum_{j=1}^{3} W_j(\boldsymbol{\xi}_j) + \varepsilon^2 \sum_{j=1}^{3} u^{(j)}(\boldsymbol{x}) + \varepsilon^2 v(\boldsymbol{x},\phi) + o(\varepsilon^2), \quad \varepsilon \to 0$$



The asymptotic ansatz

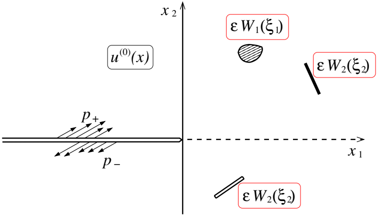
$$u(\boldsymbol{x}, \varepsilon) = \boxed{u^{(0)}(\boldsymbol{x})} + \varepsilon \sum_{j=1}^{3} W_j(\boldsymbol{\xi}_j) + \varepsilon^2 \sum_{j=1}^{3} u^{(j)}(\boldsymbol{x}) + \varepsilon^2 v(\boldsymbol{x}, \phi) + o(\varepsilon^2), \quad \varepsilon \to 0$$



1. Solution of the unperturbed problem ($\varepsilon = 0$)

The asymptotic ansatz

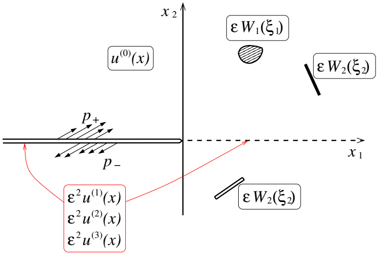
$$u(\boldsymbol{x},\varepsilon) = u^{(0)}(\boldsymbol{x}) + \boxed{\varepsilon \sum_{j=1}^{3} W_j(\boldsymbol{\xi}_j)} + \varepsilon^2 \sum_{j=1}^{3} u^{(j)}(\boldsymbol{x}) + \varepsilon^2 v(\boldsymbol{x},\phi) + o(\varepsilon^2), \quad \varepsilon \to 0$$



1. Solution of the unperturbed problem ($\varepsilon = 0$)
2. Boundary layers concentrated near the defects

The asymptotic ansatz

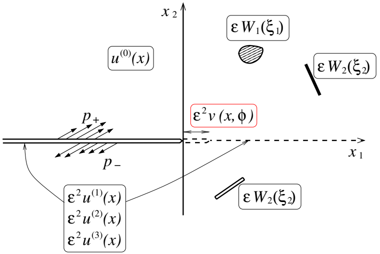
$$u(\boldsymbol{x}, \varepsilon) = u^{(0)}(\boldsymbol{x}) + \varepsilon \sum_{j=1}^{3} W_j(\boldsymbol{\xi}_j) + \boxed{\varepsilon^2 \sum_{j=1}^{3} u^{(j)}(\boldsymbol{x})} + \varepsilon^2 v(\boldsymbol{x}, \phi) + o(\varepsilon^2), \quad \varepsilon \to 0$$



1. Solution of the unperturbed problem ($\varepsilon = 0$)
2. Boundary layers concentrated near the defects
3. Additional terms to adjust the BC and IC disturbed by the boundary layers

The asymptotic ansatz

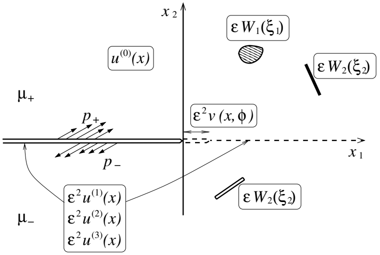
$$u(\boldsymbol{x},\varepsilon) = u^{(0)}(\boldsymbol{x}) + \varepsilon \sum_{j=1}^{3} W_j(\boldsymbol{\xi}_j) + \varepsilon^2 \sum_{j=1}^{3} u^{(j)}(\boldsymbol{x}) + \boxed{\varepsilon^2 v(\boldsymbol{x},\phi)} + o(\varepsilon^2), \quad \varepsilon \to 0$$



1. Solution of the unperturbed problem ($\varepsilon = 0$)
2. Boundary layers concentrated near the defects
3. Additional terms to adjust the BC and IC disturbed by the boundary layers
4. Perturbation associated with the crack advance $\varepsilon^2\phi$

The asymptotic ansatz

$$u(\boldsymbol{x},\varepsilon) = \ u^{(0)}(\boldsymbol{x}) \ + \ \varepsilon \sum_{j=1}^{3} W_j(\boldsymbol{\xi}_j) \ + \ \varepsilon^2 \sum_{j=1}^{3} u^{(j)}(\boldsymbol{x}) \ + \ \varepsilon^2 v(\boldsymbol{x},\phi) \ + o(\varepsilon^2), \quad \varepsilon \to 0$$



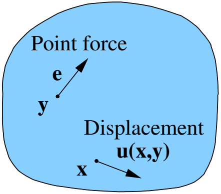
1. Solution of the unperturbed problem ($\varepsilon = 0$)
2. Boundary layers concentrated near the defects
3. Additional terms to adjust the BC and IC disturbed by the boundary layers
4. Perturbation associated with the crack advance $\varepsilon^2\phi$

- Using the linearity of the problem, we analyse the perturbation of each defect separately (superposition principle).
- The method can be extended to a finite number of defects, provided that the distance between defects remains finite (composite is dilute).

The first challenge: Weight functions for an interfacial crack

"Weight functions are fundamental solutions for the problem of a cracked body"

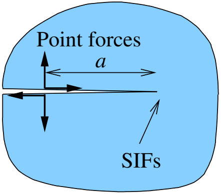
Linear elasticity:



Green's function for elastic body

$$u_i(\boldsymbol{x},\boldsymbol{y}) = G_{ij}(\boldsymbol{x},\boldsymbol{y})e_j$$

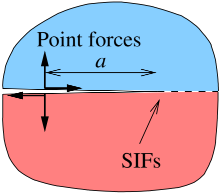
Linear fracture mechanics:



Weight functions for cracked homogeneous body

$$K = \sqrt{\frac{2}{\pi}}(1+i)a^{-1/2}$$

Interfacial crack:



Weight functions for interfacial crack

?

The weight functions for an interfacial crack

- Mode III symmetric and skew-symmetric weight functions:

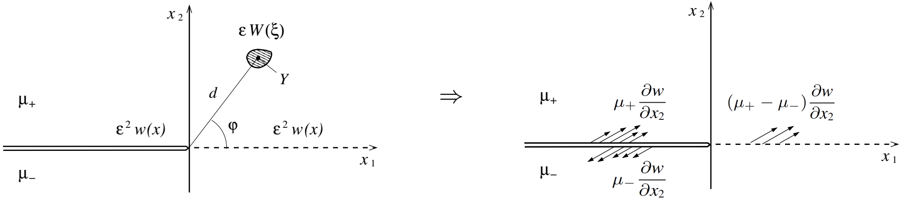
$$[\![U_3]\!](x_1) = \begin{cases} \dfrac{1-i}{\sqrt{2\pi}} x_1^{-1/2}, & \text{for } x_1 > 0, \\ 0, & \text{for } x_1 < 0, \end{cases} \qquad \langle U_3 \rangle = \frac{\eta}{2} [\![U_3]\!],$$

- Integral formula for the computation of the Mode III SIF:

$$K_{\mathrm{III}} = -(1+i) \lim_{x_1' \to 0+} \int_{-\infty}^{0} \{ \underbrace{[\![U_3]\!](x_1' - x_1) \langle p_3 \rangle (x_1)}_{\text{symmetric part}} + \underbrace{\langle U_3 \rangle (x_1' - x_1) [\![p_3]\!](x_1)}_{\text{skew-symmetric part}} \} dx_1$$

$$\eta = \frac{\mu_- - \mu_+}{\mu_- + \mu_+} \qquad \text{bimaterial parameter}$$

The second challenge: singular perturbation for a small defect



The defect is replaced by "effective" tractions

Dipole field:

$$\varepsilon^2 w(\boldsymbol{x}) = -\frac{\varepsilon^2}{2\pi} \left[\nabla u^{(0)} \Big|_{\boldsymbol{Y}} \right] \cdot \left[\mathcal{M} \frac{\boldsymbol{x} - \boldsymbol{Y}}{|\boldsymbol{x} - \boldsymbol{Y}|^2} \right] + o(\varepsilon^2), \quad \varepsilon \to 0$$

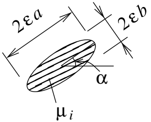
$\mathcal{M}$ is the dipole matrix

Perturbation of the SIF:

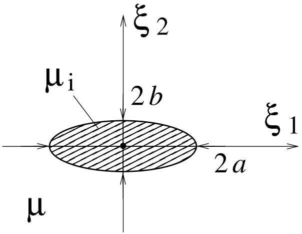
$$\Delta K_{\mathrm{III}} = -\sqrt{\frac{2}{\pi}} \frac{\mu_+ \mu_-}{\mu_+ + \mu_-} \nabla u^{(0)} \Big|_{\boldsymbol{Y}} \cdot \mathcal{M} \boldsymbol{c}$$

$$\boldsymbol{c} = \frac{1}{2d^{3/2}} \left[-\sin \frac{3\varphi}{2}, \cos \frac{3\varphi}{2} \right]$$

Example: Elliptic elastic inclusion



$$\Rightarrow \quad \boldsymbol{\xi} = \frac{\boldsymbol{x} - \boldsymbol{Y}}{\varepsilon} \quad \Rightarrow$$



Dipole field:

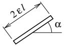
$$\varepsilon^2 w(\boldsymbol{x}) = -\frac{\varepsilon^2}{2\pi} \left[\nabla u^{(0)} \Big|_{\boldsymbol{Y}} \right] \cdot \left[\mathcal{M} \frac{\boldsymbol{x} - \boldsymbol{Y}}{|\boldsymbol{x} - \boldsymbol{Y}|^2} \right] + o(\varepsilon^2), \quad \varepsilon \to 0$$

Dipole matrix

$$\mathcal{M} = -\frac{\pi}{2} ab(1+e)(\mu_\star - 1) \begin{bmatrix} \dfrac{1+\cos 2\alpha}{e+\mu_\star} + \dfrac{1-\cos 2\alpha}{1+e\mu_\star} & -\dfrac{(1-e)(\mu_\star - 1)\sin 2\alpha}{(e+\mu_\star)(1+e\mu_\star)} \\ -\dfrac{(1-e)(\mu_\star - 1)\sin 2\alpha}{(e+\mu_\star)(1+e\mu_\star)} & \dfrac{1-\cos 2\alpha_1}{e+\mu_\star} + \dfrac{1+\cos 2\alpha}{1+e\mu_\star} \end{bmatrix}$$

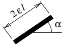
$$\mu_\star = \mu/\mu_i \qquad e = b/a$$

Dipole matrix for different types of defects



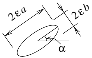
microcrack

$$\boldsymbol{\mathcal{M}} = -\pi l^2 \begin{bmatrix} \sin^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \cos^2\alpha \end{bmatrix}$$



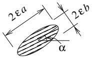
rigid line inclusion

$$\boldsymbol{\mathcal{M}} = \pi l^2 \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}$$



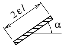
elliptic void

$$\boldsymbol{\mathcal{M}} = -\tfrac{\pi}{2} ab(1/e+1) \begin{bmatrix} 1-\cos 2\alpha + e(1+\cos 2\alpha) & -(1-e)\sin 2\alpha \\ -(1-e)\sin 2\alpha & 1+\cos 2\alpha + e(1-\cos 2\alpha) \end{bmatrix}$$



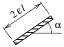
elliptic rigid inclusion

$$\boldsymbol{\mathcal{M}} = \tfrac{\pi}{2} ab(1/e+1) \begin{bmatrix} 1+\cos 2\alpha + e(1-\cos 2\alpha) & (1-e)\sin 2\alpha \\ (1-e)\sin 2\alpha & 1-\cos 2\alpha + e(1+\cos 2\alpha) \end{bmatrix}$$



line defect (soft)

$$\boldsymbol{\mathcal{M}} = -\tfrac{\pi}{2}\,\frac{l^2}{\kappa l+1} \begin{bmatrix} 1-\cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & 1+\cos 2\alpha \end{bmatrix}$$



line defect (stiff)

$$\boldsymbol{\mathcal{M}} = \tfrac{\pi}{2}\,\frac{\kappa l^2}{\kappa+l} \begin{bmatrix} 1+\cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & 1-\cos 2\alpha \end{bmatrix}$$

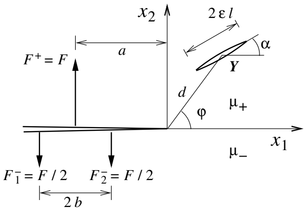
Application: Shielding and amplification effects

Definition:

$$\Delta K_{\text{III}} = -\sqrt{\frac{2}{\pi}} \frac{\mu_+ \mu_-}{\mu_+ + \mu_-} \nabla u^{(0)}\Big|_{\boldsymbol{Y}} \cdot \boldsymbol{\mathcal{M}} \boldsymbol{c}$$

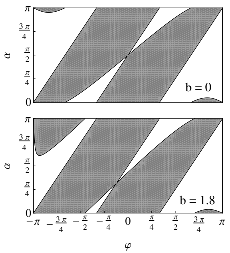
$\Delta K_{\text{III}} < 0$:	*shielding effect*	"the defect is *preventing* the propagation"
$\Delta K_{\text{III}} > 0$:	*amplification effect*	"the defect is *promoting* the propagation"
$\Delta K_{\text{III}} = 0$:	*neutral*	"the defect has no effect"

Example: shielding/amplification diagrams for macro-microcrack interaction



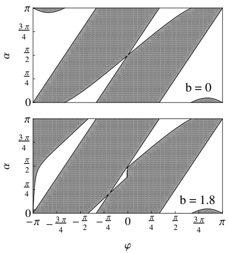
$d = 1$, $2\varepsilon l = 0.02$, $a = 2$

Homogeneous plane:



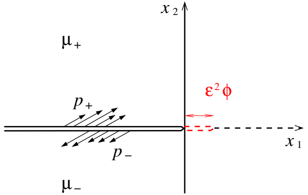
$\mu_+ = \mu_-$

Bimaterial plane:



$\mu_+ / \mu_- = 0.1$

The third challenge: Singular perturbation for crack advance



The Betti identity:

$$\int_{-\infty}^{\infty}\left\{[\![U]\!](x_1'-x_1)\langle\sigma\rangle(x_1)+\langle U\rangle(x_1'-x_1)[\![\sigma]\!](x_1)-\langle\Sigma\rangle(x_1'-x_1)[\![u]\!](x_1)\right\}dx_1=0$$

u, σ physical solution $\qquad$ U, Σ weight functions

Perturbation of the SIF: $\qquad \varepsilon^2\Delta K_{\rm III}^{\phi}=\dfrac{\varepsilon^2\phi}{2}A_{\rm III}^{(0)}$

$$A_{\rm III}^{(0)}=\sqrt{\frac{2}{\pi}}\int_{-\infty}^{0}\left(\langle p\rangle(x_1)+\frac{\eta}{2}[\![p]\!](x_1)\right)(-x_1)^{-3/2}dx_1$$

Analysis of a stable quasi-static propagation

The stress intensity factor is expanded as follows:

$$K_{\text{III}} = K_{\text{III}}^{(0)} + \varepsilon^2 \left(\Delta K_{\text{III}}^{\phi} + \sum_{j=1}^{3} \Delta K_{\text{III}}^{(j)} \right) + o(\varepsilon^2), \quad \varepsilon \to 0$$

$\Delta K_{\text{III}}^{\phi} = \dfrac{\varepsilon^2 \phi}{2} A_{\text{III}}^{(0)}$: perturbation produced by the elongation of the crack along the interface

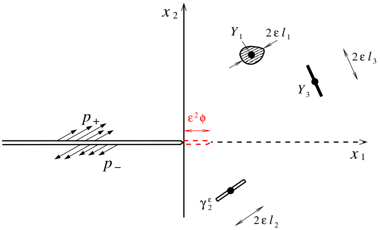
$\sum_{j=1}^{3} \Delta K_{\text{III}}^{(j)}$: perturbation produced by the defects

We assume that the crack propagation is stable and quasi-static: $G = G_c$

$$G = \frac{1}{4}\left(\frac{1}{\mu_+} + \frac{1}{\mu_-}\right) K_{\text{III}}^2 \quad \Rightarrow \quad \Delta K_{\text{III}}^{\phi} + \sum_{j=1}^{3} \Delta K_{\text{III}}^{(j)} = 0 \quad \Rightarrow \quad \boxed{\phi = \frac{2}{A_{\text{III}}^{(0)}} \sum_{j=1}^{3} \Delta K_{\text{III}}^{(j)}}$$

$$\Delta K_{\text{III}}^{(j)} = -\sqrt{\frac{2}{\pi}} \frac{\mu_+ \mu_-}{\mu_+ + \mu_-} \nabla u^{(0)} \Big|_{\boldsymbol{Y}_j} \cdot \boldsymbol{\mathcal{M}}_j \boldsymbol{c}_j, \qquad \boldsymbol{c}_j = \frac{1}{2 d_j^{3/2}} \left[-\sin \frac{3\varphi_j}{2}, \cos \frac{3\varphi_j}{2} \right]$$

Application: Crack propagation and arrest



Given a configuration of defects and position of the crack tip, the incremental crack advance ϕ is given by:

$$\phi = \frac{2}{A_{\mathrm{III}}^{(0)}} \sum_{j=1}^{3} \Delta K_{\mathrm{III}}^{(j)}$$

It is possible to update the configuration with the new position of the crack tip and recompute the incremental crack advance in the new configuration, following an iterative procedure:

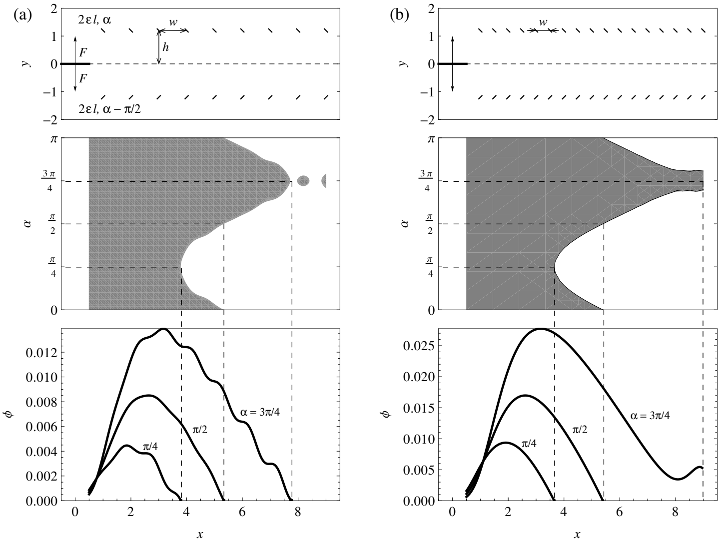
- the crack "accelerates" when the increment ϕ is increasing
- the crack "decelerates" when the increment ϕ is decreasing
- the crack "arrests" when a neutral configuration is reached ($\phi = 0$)

The total crack elongation is computed as:

$$x(N) = \sum_{i=0}^{N} \varepsilon^2 \phi_i$$

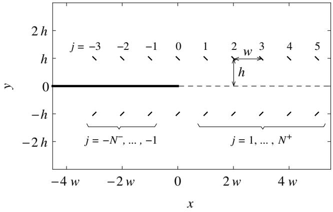
where N is the number of iterations.

Crack propagation in a finite array of defects



Conclusions and possible extensions

This asymptotic method can be extended to the case of an infinite array of defects



$$\phi = \frac{al^2 \cos 2\alpha}{2h^2} \left\{ -1 + \sum_{j=1}^{N^+} \frac{\left(1 - \frac{h^2}{j^2 w^2}\right) \frac{h^2}{j^2 w^2}}{\left(1 + \frac{h^2}{j^2 w^2}\right)^2} + \sum_{j=1}^{N^-} \frac{\left(1 - \frac{h^2}{j^2 w^2}\right) \frac{h^2}{j^2 w^2}}{\left(1 + \frac{h^2}{j^2 w^2}\right)^2} \right\}$$

Take the limit as $N^+ \to \infty$:

$$\phi = \frac{al^2 \cos 2\alpha}{2h^2} \left\{ -\frac{1}{2} - \frac{\left(\frac{\pi h}{w}\right)^2}{2 \sinh^2\left(\frac{\pi h}{w}\right)} + \sum_{j=1}^{N^-} \frac{\left(1 - \frac{h^2}{j^2 w^2}\right) \frac{h^2}{j^2 w^2}}{\left(1 + \frac{h^2}{j^2 w^2}\right)^2} \right\}$$

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